The Brownian Web

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Part C Dissertation - March 2008
Acknowledgements

I am extremely grateful to my supervisors, Alison Etheridge and Pierre Tarrès, for guiding me through the challenging material on which this dissertation is based and for their patient explanations of mathematical details. I wish to thank R. Arratia for making a seminal unpublished manuscript [1] available to me. I am very grateful to R. Sun for helpful comments (in particular regarding [30, 31]) and explanation of a detail in [12].

I am, moreover, very grateful to Arend Janssen for reading a draft of this dissertation and to my parents, Angela and Hans Mattauch, for proof-reading and encouragement. Special mention goes to David Yadin for constant support and to Glenys Luke for giving wise advice whenever needed. Finally I wish to thank Robert Goudie and Tsubasa Itani for some help with L\LaTeX\ and everyone else who contributed to a fantastic time as an undergraduate at St. Hugh’s College.

Linus Mattauch
March 2008

Note concerning the proposal

I wish to note that my study of the Brownian web turned out to be somewhat less connected to the theory of interacting particle systems than I anticipated at the time of submitting the proposal. So the connection between the voter model and coalescing random walks is included in this dissertation (as Appendix A), but the focus of the present work is exclusively on the Brownian web itself. An alternative for the end of the project was stated in the proposal: I decided to include the topic of the “Poisson tree” (not that of self-interacting random walks) into this dissertation and also tailored my account of the Brownian web accordingly.
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Chapter 1
Introduction and preliminary results

1.1 Introduction

This dissertation is devoted to the study of a remarkable stochastic object that has recently been subject to much research: The Brownian web is the collection of one-dimensional coalescing Brownian motions starting from every point in space and time, \( \mathbb{R} \times \mathbb{R} \).

It is a well-known fact that (under suitable rescaling and in the sense of weak convergence – to be made precise later) a simple random walk converges to a Brownian motion. In the same sense the Brownian web arises as a scaling limit of the simple coalescing random walks on \( \mathbb{Z} \times \mathbb{Z} \).

Much of the work set out below concerns developing the right framework in which to obtain the Brownian web as a limiting object in the sense of weak convergence. That framework was introduced by Fontes et al. [12], which supersedes the viewpoint of Arratia ([1], who studied the Brownian web first), which we shall also briefly consider.

We present several recent results about the convergence of simple and non-simple random walks, as well as of the “Poisson tree” to the Brownian web in that framework (partly verifying claims where no proofs were given in the literature).

Moreover, in Section 3.5, we discuss an example of a random configuration of paths of which convergence to the Brownian web, to the best of our knowledge, has not been considered in the literature and we prove steps towards showing that convergence might hold.

To introduce some key ideas of the subject of this work heuristically, let us start by considering random paths in the plane as follows (cf. [12]). Let \( E := \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : i + j \text{ even}\} \). We think of \( i \) as the space and \( j \) as the time coordinate. Let \( (Y_{i,j})_{(i,j) \in E} \) be a collection of independent identically distributed (i.i.d.) random variables with

\[
\mathbb{P}(Y_{i,j} = +1) = \mathbb{P}(Y_{i,j} = -1) = \frac{1}{2}.
\]

Imagine an infinite number of random walkers, starting from all points of \( E \) and
moving left or right according to the value of $Y_{i,j}$. It follows that if two walkers meet at a space-time point they coalesce – that is, they “move together” from then on.

If the increments $Y_{i,j}$ remain i.i.d. but may take values different from $-1, 1$ with non-zero probability, so that the random walks defined from them become non-simple, their paths may cross each other, and only coalesce if they meet at a point of the integer lattice, a case we will consider later.

In the case of simple random walks it is easy to see that two of them, one starting from $(i, j)$, $i + j$ even, and one from $(k, l)$, $k + l$ odd, would never coalesce: Their “distance” $d$ in discrete space-time (for $(s, u), (v, w) \in \mathbb{Z} \times \mathbb{Z}, d((s, u), (v, w)) = |s - v| + |u - w|$) is an odd number after any number of jumps, thus never 0. So one restricts to one of the two independently evolving copies of a collection of random walks, those with “even” starting points in our case. Define the collection of simple random walks $Z_{i_0,j_0}$ starting at every $(i_0, j_0) \in E$ by

$$Z_{i_0,j_0}(j + 1) = Z_{i_0,j_0}(j) + Y_{Z_{i_0,j_0}(j),j} \quad Z(j_0) = i_0 \quad \text{for } j \geq j_0.$$ 

As we will eventually need to consider continuous time, we implicitly think of the coalescing random walks as the paths resulting from linearly interpolating the variables just defined (Figure 1). In this dissertation we shall always take “time” to be on the vertical coordinate axis.

One can define coalescing random walks in continuous time similarly (see for instance [30, p. 3]), with one walker starting from every point of $\mathbb{Z} \times \mathbb{R}$.

It is well-known that a simple random walk converges to a Brownian motion if one lets the size of its steps tend to 0 and time and space are suitably rescaled. More precisely, Donsker’s Invariance Principle states that if one rescales spatial steps
by size $\delta$ and time steps by size $\delta^2$ then as $\delta \to 0$ the distribution of the random walk converges in the space of continuous paths to the standard Wiener measure (see Section 1.2). For finitely many coalescing random walks, independent before collision, analogous results hold.

The rationale behind studying the Brownian web is hence the analogy with Donsker’s Invariance Principle. Let $\delta = \frac{1}{n}$ and let $X_{1/n} = \{Z^n_{i_0,j_0} : (i_0,j_0) \in E\}$ denote the collection of all rescaled coalescing random walks with the steps having size $(\frac{1}{n}, \frac{1}{n^2})$. One would like to study the limiting object as $n \to \infty$, but the difficulty is to define a metric space in which the limit can be made precise. One chooses to consider the $X_{1/n}$ to be random variables into a space of compact sets of paths and calls their limit the Brownian web, which turns out to be the collection of coalescing Brownian motions starting from every even space-time point.

Constructing the Brownian web as a random variable into a space of sets of paths should consequently be thought of as somewhat similar, at least in purpose, to Wiener’s construction and characterisation of Brownian motion as a random variable into a paths space. A convergence theorem stating that the coalescing simple random walks tend to the Brownian web (Section 3.1) plays much the same role as Donsker’s Invariance Principle in the case of a single random walk. Once an appropriate space and topology have been introduced, it is however easier not to define the Brownian web as a limiting object of the coalescing random walks, but rather as a closure (in the appropriate path space) of all coalescing Brownian motions starting at rational space-time points.

From the above description, the mathematical importance of the Brownian web should be evident: one may hope that the Brownian web (in analogy with Brownian motion) will turn out to be a fairly universal scaling limit in the future (as the examples in Chapter 3 suggest). It is already now a useful object in a variety of probabilistic contexts, not just that of coalescing random walks. It has also been studied in the context of self-interacting processes, coalescing-branching configurations and stochastic flows.

The Brownian web was first investigated by Arratia [1, 2] about 30 years ago, motivated by asymptotic behaviour of the one-dimensional voter model. He constructed coalescing one-dimensional Brownian motions starting from $\mathbb{R}$, later extending this to starting them at every space-time point, but without the now standard sophisticated topology. Arratia noted that in letting the step size of coalescing random walks tend to 0, in the limit there occur (non-deterministic) space-time points from which multiple limit paths evolve. The collection of such points has Lebesgue measure zero. He provided conventions to avoid there being more than one path starting from some points in the limiting object. About ten years ago Tóth and Werner [32] used coalescing Brownian motions in the plane as an auxiliary means of constructing a continuous ‘self-repelling’ motion.

Most importantly for this work, motivated by concepts from statistical physics, Fontes, Isopi, Newman and Ravishankar [11, 12, 13] (who coined the term Brownian web) introduced a topological framework for weak convergence results in 2002. The motivation for the now standard topology of the Brownian web of [13] is to keep...
the multiplicities of paths and accept them as part of a certain closure of coalescing Brownian motions from a countable dense set in $\mathbb{R}^2$ as well as to be able to define convergence properly.

There is ongoing research to discover new contexts in which the Brownian web appears as a useful object, we will consider: the work of Ferrari et al. [7] which connected the Brownian web to the “Poisson tree” and that of Sun [30], who treated the case of random walks with crossing paths. Very recent publications use the Brownian web in the context of stochastic flows (which we shall not deal with). In others it is extended to a coalescing-branching situation (see [31]) or further properties of the Brownian web are revealed, see Appendix D.

This dissertation is organised as follows. In the remainder of this chapter we shall provide the necessary background for results about the Brownian web by giving a statement of Donsker’s Invariance Principle and discussing the case of finitely many coalescing Brownian motions. In the spirit of Arratia’s work [1] we shall take a first look at coalescing Brownian motions starting from every point of $\mathbb{R}^2$ and prove that for coalescing Brownian motions starting at every point of $\mathbb{R}$ at time $0$, at any time $t_0 > 0$ only discretely many distinct Brownian motions are left. We shall also take a glimpse at the broad motivation for the whole subject by looking at the connection between the voter model and coalescing random walks in Appendix A.

In Chapter 2 (following [12]) we introduce the standard topology for the Brownian web and give a construction of the Brownian web as a random variable in a space of compact sets of paths. We also give an equivalent characterisation in terms of counting random variables. We mention the dual and Double Brownian web and the classification of the points in the Brownian web with multiple paths starting from them in Appendix B, crucial to the connection of the Brownian web with self-interacting processes.

Chapter 3 – at the heart of the theory of the Brownian web and of this dissertation – deals with convergence to the object. We state and discuss a convergence theorem and prove that the simple coalescing random walks introduced above converge to the Brownian web according to this theorem. We also look at convergence in the more general case of non-simple coalescing random walks and discuss in some detail the remarkable fact that trees generated from a two-dimensional Poisson process converge to the Brownian web as well. This incorporates material from [8, 12, 22, 30]. We consider a modified model of the Poisson tree and display our own work which suggests that even this modification converges to the Brownian web.

Appendix D finally summarises the most recent developments in the subject and looks at alternative approaches to the Brownian web.
1.2 Donsker’s Invariance Principle

Donsker’s Invariance Principle is a result about the limiting behaviour of a single random walk path. It is intuitive that in some sense a simple random walk converges, as its increments get smaller and more frequent (and under some rescaling) to a Brownian motion. The proper sense for such a result is convergence in “distribution”; Donsker’s Invariance Principle states, informally, that in distribution a random walk converges to a Brownian motion. It is necessary to recall very briefly the concepts of weak convergence and the Wiener measure (cf. [20, 26]) to state the result formally.

Definition 1 (weak convergence, convergence in distribution) Let $E$ be a metric space with Borel-$\sigma$-algebra, $\mu, \mu_1, \mu_2, ...$ finite measures on $E$. $(\mu_n)_{n \in \mathbb{N}}$ converges weakly to $\mu$ if

$$\int f \, d\mu_n \to \int f \, d\mu \quad \text{as } n \to \infty \text{ for all } f : E \to \mathbb{R} \text{ bounded and continuous.}$$

Let $X, X_1, X_2, ...$ be random variables taking values in $E$ with distributions $\mu, \mu_1, \mu_2, ...$. Then the sequence $(X_n)_{n \in \mathbb{N}}$ converges in distribution to $X$ if the distributions $(\mu_n)_{n \in \mathbb{N}}$ converge weakly to $\mu$.

Remark: If $E$ is complete and separable, then weak limits are unique (cf. [20, p. 236-240]).

We will need the following concept later in this dissertation:

Definition 2 (tightness) A family $\mathcal{F}$ of finite measures on a metric space $E$ with Borel-$\sigma$-algebra is called tight, if for any $\epsilon > 0$ there exists a compact $K \subset E$ such that

$$\sup \{ \mu(E \setminus K) : \mu \in \mathcal{F} \} < \epsilon.$$ 

Recall that a Brownian motion can be thought of as a random variable into the space of continuous paths. Let $\Omega = C([0, \infty)) \subset \mathbb{R}^{[0, \infty)}$ and $(X_t)_{t \geq 0}$ the process given by the restriction to $\Omega$ of the canonical projection $X_t : \mathbb{R}^{[0, \infty)} \to \mathbb{R}$, $\omega \to \omega(t)$. Define a metric on $C([0, \infty))$ by letting

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} d_n(f, g) \quad \text{and} \quad d_n = \| (f - g) |_{[0,n]} \|_{\infty} \wedge 1.$$ 

One can prove that this metric is complete and gives the topology of uniform convergence on compact sets. Note that $\Omega$ is separable by Weierstrass’ Approximation Theorem, so weak limits are unique. One can also prove that $\mathcal{B}(\Omega, d) = \sigma(X_t, t \in [0, \infty)) := (\mathcal{B}(\mathbb{R}))^{\otimes [0, \infty)}|_{\Omega}$, that is the $\sigma$-algebra generated by the projections, which is by definition the product $\sigma$-algebra on the space, is the same as the Borel-$\sigma$-algebra generated from the metric. Set $\mathcal{A} = \mathcal{B}(\Omega, d)$.

Now let $B$ be a Brownian motion on any probability space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$.

Lemma 3 With respect to a probability measure $\mathbb{P} = \tilde{\mathbb{P}} \circ B^{-1}$ on $\Omega = C([0, \infty))$ the canonical process $X = (X_t)_{t \geq 0}$ is a Brownian motion.

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**Proof:** By the definition of Brownian motion there is $\Omega \in \tilde{A}$ with $\tilde{P}(\Omega) = 1$ such that $B(\omega) \in C([0, \infty))$ for any $\omega \in \Omega$. Let $\tilde{A} = \tilde{A}|_{\Omega}$, $\tilde{P} = \tilde{P}|_{\tilde{A}}$. Then $B : \Omega \rightarrow C([0, \infty))$ is still a Brownian motion and is measurable with respect to $(\tilde{A}, \mathcal{A})$. This last claim is verified as follows: As $\mathcal{A}$ is generated by the canonical projections, it is sufficient to prove that for any $D \in \mathcal{B}(\mathbb{R})$, $B^{-1}(X_i^{-1}(D)) \in \tilde{A}$. But this is true as $B^{-1}(X_i^{-1}(D)) = (X_i \circ B)^{-1}(D) = B_i^{-1}(D) \in \mathcal{A}$ where $B_t$ denotes the random variable at time $t$ of the Brownian motion. □

**Definition 4 (Wiener measure)** The Wiener measure $\mathbb{P}_W$ is the probability measure on $\Omega = C([0, \infty))$ with respect to which the canonical process $X$ is a Brownian motion. Call $(\Omega, \mathcal{A}, \mathbb{P}_W)$ the Wiener space and $X$ the canonical Brownian motion. For a Brownian motion starting at general $X_0 \equiv x \in \mathbb{R}$, let $\mathbb{P}_W^x$ be that measure on $C([0, \infty))$ for which $X = (X_t - x)_{t \geq 0}$ is a standard Brownian motion.

We are now in a position to state Donsker’s Theorem (see [20, p. 456-57]): Let $Y_1, Y_2, ...$ be i.i.d. random variables with $\mathbb{E}(Y_1) = 0$, $\text{var}(Y_1) = \sigma^2 > 0$. For $t > 0$, let

$$S^n_t = \sum_{i=1}^{\lfloor tn \rfloor} Y_i, \quad \hat{S}_t^n = \frac{1}{\sqrt{\sigma^2 n}} S^n_t.$$  

By the central limit theorem (denoting the distribution of a random variable $X$ by $\mathcal{L}(X)$), in distribution, $\mathcal{L}(\hat{S}_t^n) \rightarrow \mathcal{N}_{0,t}$.

However, a much stronger conclusion is true. If we take a linearly interpolated version

$$\bar{S}_t^n = \frac{1}{\sqrt{\sigma^2 n}} \sum_{i=1}^{\lfloor nt \rfloor} Y_i + \frac{(tn - \lfloor nt \rfloor)}{\sqrt{\sigma^2 n}} Y_{\lfloor nt \rfloor + 1}$$

then we have

**Theorem 5 (Donsker’s Invariance Principle)** In the sense of weak convergence on $C([0, \infty))$, the distributions of the paths $\bar{S}_t^n = (\bar{S}_t^n)_{t \in [0, \infty)}$ converge to the Wiener measure $\mathbb{P}_W$, that is

$$\mathcal{L}(\bar{S}_t^n) \rightarrow \mathbb{P}_W \text{ as } n \rightarrow \infty.$$  

**Proof:** Beyond the purpose of this dissertation, but see for instance [20, p. 457-459] or [26]. □

A version of Donsker’s Invariance Principle is true for continuous-time random walks.

Let us now explain the rescaling of the random walks stated in the introduction: If every of the i.i.d. increments $Y_j$ is such that $\mathbb{E}(Y_j) = 0$, $\text{var}(Y_j) = 1$, then rescaling $Y_j$ to steps of size $\delta$ in space, $\delta^2$ in time yields still zero expectation, but $\text{var}(Y_j) = \delta^2$. Now let $\delta = \frac{1}{m}$ and consider the above with $n_m = m^2$ because the time step size is now $\frac{1}{m^2}$. Then up to time $t$ we take the sum over $[m^2 t]$ random variables. Of course, if $\mathcal{L}(S^n_t) \rightarrow \mathbb{P}_W$, then $\mathcal{L}(S_{nm}^n) \rightarrow \mathbb{P}_W$. Note that

$$\frac{1}{\sqrt{\delta^2 n}} S_t^n = \frac{1}{\sqrt{\frac{1}{m^2}} n^2} S_{tm}^n = S_{tm}^n$$

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if $n = m^2$, and so $S_t^{m^2} = S_t^{m^2} + (tm^2 - |m^2t|)Y_{|m^2t|+1} \to \mathbb{P}_W$ as $m \to \infty$ weakly. Therefore a random walk, when rescaled by $(\frac{1}{m}, \frac{1}{m^2})$, converges to a Brownian motion in distribution.

1.3 Arratia on the Brownian web: Finitely many coalescing Brownian motions and coalescing Brownian motions from all space points

In this section we provide a brief summary of Arratia’s groundbreaking work on the Brownian web, based on material from [1, 2]. Having the definitive result at hand for the case of a single Brownian motion as the limiting object of a random walk, the next step is to look at the case of finitely many coalescing Brownian motions, all starting at the same time. What is at issue is whether one can, starting with $n$ independent Brownian motions, construct $n$ coalescing Brownian motions such that their joint distribution is – regardless of the way we define coalescence – always the same. By appealing to the strong Markov property, one can show that any collision rule that does not depend on the realisation of the Brownian motions after the time of coalescence will yield the same object in distribution.

The main way Arratia thinks of finitely many coalescing Brownian motions is as a system of $n$ distinct indestructible particles that stick together after collision. We would like to construct a system $Z$ of $n$ distinct particles as a single point in $\mathbb{R}^n$ as follows: label the particles in order of their initial position from left to right, i.e. $Z(0)$ is such that $Z_1(0) < Z_2(0) < \ldots < Z_n(0)$. So, one would like to have that $Z_i(t) \leq \ldots \leq Z_n(t)$ and that a coalescing property holds: For $s \geq t \geq 0$, $1 \leq i,j \leq n$, $Z_i(t) = Z_j(t)$ implies $Z_i(s) = Z_j(s)$. Now given any $n$ independent paths, one can construct such a system of $n$ coalescing paths. Assign the particles a ranking $R$, via permuting them by $r \in S^n$ say, and figuratively speaking “each particle would like to follow the path that it starts from, but after colliding with a particle of higher rank, it must follow that particle along its preferred path” [2, p. 27]. One makes this precise by defining a collision rule as a mapping $K_r$ on an appropriate path space. Arratia can show that each of the possible $n!$ rankings gives the same object in joint distribution if the paths are that of one-dimensional coalescing Brownian motion (see [1, p. 12 ff.] for the details in some more generality). Using that the collision rule commutes in a suitable way with time shifts, it is proved that, if each of the $n$ independently evolving paths is a Markov process on $\mathbb{R}$ and the corresponding product Markov process has the strong Markov property, then all $n!$ differently defined coalescing systems have the same distribution, see ([1, p. 15]).

Arratia then constructs the Brownian web as the collection of all Brownian motions starting everywhere on $\mathbb{R}$ at any time and independent before coalescing upon collisions, taking the movement of each particle to be a stochastic process (cf. [1, p. 16-19]).

Let $c_{x,t}^s$ (omit $s$ if $s = 0$) be a random variable representing the position at time $t \geq s$, of a particle which starts from site $x$ at time $s$. We temporarily define the
Brownian web in an elementary way:

**Definition 6 (Brownian web, after Arratia [1])** Let \( c^x_{s,t} \) be as above and consider each path \( \{c^x_{s,u} : u \geq s\} \) to be an element of \( C([0,\infty)) \), with uniform topology on compact intervals, given by distance \( d \). Then \( c = \{c^x_{s,t} : \text{ for all } x, s, t \in \mathbb{R}, s \geq t \geq 0 \} \) is the Brownian web, and is considered a stochastic process with parameter set in \( \mathbb{R}^2 \) taking values in \( C([0,\infty)) \), if it satisfies the following: for each \( x, s, t \in \mathbb{R}, c^x_{s,0} = x \) and the finite-(n)-dimensional distributions of the process are those of the result of applying an appropriate collision rule to \( n \) independent Brownian motions starting from the specified \( n \) different space-time starting points. (For a formal version see [1, p.16-17]).

Note that Arratia can then conclude by Kolmogorov’s extension theorem (cf. e.g. [24, p. 11]) that there is such a process in \( C([0,\infty])^{\mathbb{R}\times\mathbb{R}} \) having these finite-dimensional distributions.

The coalescing property is in this case:

\[
\forall \omega \in \Omega, \quad s, t \geq 0, \quad c^0_s = x, c^0_t = y, \quad \text{we have that } c^x_s = c^y_t \implies c^x_{s+t} = c^y_{s+t}.
\]

Hence, as we consider Brownian motion as a random variable on the space \( C[0,\infty] \) – so any sample path is continuous, we have that for \( x, y \in \mathbb{R} \) with \( x < y \) that \( c^x_t \leq c^y_t \) for all \( t \). So in particular this holds for \( x, y \in \mathbb{Q} \).

We will prove the following remarkable and crucial property (following [1]) that helps understanding the Brownian web: Fix \( s \) and let

\[
c_t = \{c^x_{s,t} \text{ for all } x \in \mathbb{R}\}
\]

be the set of positions occupied in \( \mathbb{R} \) at time \( t \) by all particles starting at time \( s \). Then for any \( t > s \), \( c_t \) is discrete with probability 1. That is, almost surely almost all of the Brownian motions coalesce immediately upon coming into existence and only discretely many are left. As in that context we are only dealing with coalescing Brownian motions starting all at the same time, we may w.l.o.g. let \( s = 0 \).

Let \( c^0_t = \{c^0_x : x \in \mathbb{Q}\} \) be the random set of sites occupied at time \( t \) by particles starting on the rationals and let \( D_t \subseteq \mathbb{R} \) be the random set of division points between particles (started from the rationals) which have not coalesced by time \( t \).

\[
D_t = \{y \in \mathbb{R} | \forall x, z \in \mathbb{Q}, \text{ if } x < y \leq z \text{ then } c^x_t < c^y_t\}
\]

**Theorem 7** For \( t > 0 \), \( \{c^x_t : x \in \mathbb{R}\} \) is discrete a.s., and even \( \mathbb{P}(\{c^x_t : x \in \mathbb{R}\} \text{ is discrete } \forall t > 0) = 1 \).

**Proof** (after Arratia [1, p. 20-21], who however attributes the following argument to T. Harris; [2] contains a more complicated argument): It is sufficient to prove the two properties for \( D_t \). This is because we will use this to prove that at time \( t \) the set \( c^0_t = \{c^0_x : x \in \mathbb{Q}\} \) is also discrete. As at time \( t \) we have for \( y \in \{c^x_t : x \in \mathbb{R}\} \) that \( y \in c^0_t \) or \( y = c^z_t, z \in D_t \), the conclusion will follow.

We also need the following estimate, which we will calculate in detail later (in the proof of Theorem 24). If \( B^0, B^x \) are two independent Brownian motions starting from
0 and \( x > 0 \) respectively, then we have \( \mathbb{P}(c_t^i \neq c_t^n) = \mathbb{P}(B_s^0 < B_s^c \text{ for all } s \in [0, t]) < \frac{x}{\sqrt{\pi t}} = O(x) \) as \( x \to 0 \).

Fix \( t > 0 \). Consider a family of events as follows

\[
E_n^i = \{ c_t^{2^{-n}} < c_t^{(i+1)2^{-n}} \}, \quad n \in \mathbb{N}, 0 \leq i < 2^n
\]

and let \( N_n = \sum_{0 < t < 2^n} 1_{E_n^i} \). So for the particles on the \( n^{th} \) level of the dyadic rationals in \( [0, 1] \), \( N_n \) counts the number of adjacent pairs which have not coalesced at time \( t \). Note that by the above estimate \( \mathbb{E}(1_t) < 2^{-n}\frac{1}{\sqrt{\pi t}} \), so that \( \mathbb{E}(N_n) < \frac{1}{\sqrt{\pi t}} \) (independent of \( n \)).

For each \( \omega \), \( N_n \) increases up to \( |D_t \cap (0, 1]| \) as \( n \to \infty \). Hence \( \mathbb{E}[|D_t \cap (0, 1)|] \leq (\pi t)^{-\frac{1}{2}} \), so a.s. \( |D_t \cap (0, 1)| \) is finite. The argument did not depend on \([0, 1]\), so a.s. \( D_t \) is discrete. By the coalescence property, \( 0 < s < t \) implies \( D_t \subseteq D_s \) and so by downward monotone convergence \( \mathbb{P}(D_t \text{ is discrete } \forall t > 0) = 1 \).

Finally, we turn to \( c_t^Q \): fix \( a, b \) and take \( 0 < r < s \). Let

\[
E_n = \{ \forall t \in [r, s], c_t^{2^{-n}} < a \text{ and } c_t^{2^n} > b \}.
\]

If \( E_n \) happens, then \( \forall t \in [r, s], |c_t^Q \cap (a, b)| \leq |D_t \cap (-n, n)| \leq |D_r \cap (-n, n)| = 1 \). But we know \( \mathbb{P}(E_n \text{ e.v.}) = 1 \) and by above \( \mathbb{P}(|D_r \cap (-n, n)| < \infty) = 1 \). So it follows that \( \mathbb{P}(\forall t \in [r, s], |c_t^Q \cap (a, b)| < \infty) = 1 \) and therefore \( \mathbb{P}(\forall t > 0, c_t^Q \text{ is discrete } ) = 1 \).

In fact, the cunning definition of \( D_t \) does not only allow for a relatively straightforward proof of this key property of the Brownian web, but it also indicates the main technical problem with the object. For Arratia a problem arose ([1, p. 21 ff.]) because the distribution of \( X_t = \{ c_t^x : x \in \mathbb{R} \} \) (\( t > 0 \)) is not determined by the finite-dimensional distributions for the Brownian web \( c \) as given above. One does not pick out a unique object just through specifying the finite-dimensional distributions of \( c \). Arratia provided regularity requirements to construct a version of the Brownian web in which this is not so.

To be more precise, note that for \( t > 0 \), if

\[
c_t^{x^+} = \inf\{ c_t^q : q \in \mathbb{Q}, q \geq x \} = \lim_{q \in \mathbb{Q}, q \to x^+} c_t^q \quad \text{and}
\]

\[
c_t^{x^-} = \sup\{ c_t^q : q \in \mathbb{Q}, q \leq x \} = \lim_{q \in \mathbb{Q}, q \to x^-} c_t^q,
\]

then \( c_t^{x^+} \neq c_t^{x^-} \) if and only if \( x \in D_t \) (by definition of \( D_t \)).

Arratia shows that there can be versions of \( c \) for which \( X_t \) is differently distributed (see [1, p.22 - 23]). To get around this, he introduces the additional requirement for the Brownian web that for any \( x \in \mathbb{R} \), \( c_{s+t}^x = c_{s,t}^{x^+} \), so \( c := \{ c_{s,t}^{x^+} : \text{ for all } s, x \in \mathbb{R} \text{ fixed, all } t \geq s \in \mathbb{R} \}, \) and so fixes a particular version of \( c \).

From the next chapter on we shall adopt the contemporary viewpoint on the Brownian web, which characterises the object as the closure in a path space of the coalescing Brownian motions starting from \( \mathbb{Q}^2 \). Then a problem arises because there are random space-time points from which there are several paths starting, as shows the characterisation of \( D_t \). In that case Arratia’s extra condition prescribes which of the several paths to include and which to leave out. By contrast, the approach in the
next chapter is based on accepting the non-uniqueness of paths starting at random points. It allows a multi-valued mapping from space-time points to paths starting there, we will take the Brownian web as a whole as a random variable into a space of compact sets of paths, and show that it then has a unique distribution.
Chapter 2

Construction and characterisation of the Brownian web

We will in this chapter construct the Brownian web as a random variable into a space of compact sets of paths and discuss its properties. The main conceptual difficulty is of course that there are uncountably many space-time points, so an uncountable number of processes to construct consistently. So one treats the Brownian web in the setting below as determined by coalescing Brownian motions starting from a countable dense set in $\mathbb{R}^2$, and one takes the closure to obtain all other paths. This is the key feature exploited repeatedly in the proofs in this chapter.

The construction accomplished, we conclude the characterisation of the Brownian web by examining two other aspects in Appendix A: The idea of a dual and double Brownian web anticipated by Arratia in [2] and made possible in the framework below through a result of Soucaliuc, Tóth and Werner in [28] and a classification of the number of paths through any single point of the Brownian web. The material in this chapter is based on an article by Fontes, Isopi, Newman and Ravishankar [12].

2.1 A topology for the Brownian web

The key to the construction of the Brownian web is to choose an appropriate space, which is complete and separable in order to have unique weak limits. The space of compact sets of paths we are about to define has these properties and can be understood as an analogue of the space of continuous paths for Brownian motion discussed in Section 1.2.

We define three metric spaces as follows:

$(\mathbb{R}^2, \rho)$ a space of points in space-time

$(\Pi, d)$ a space of paths with specified starting points

$(H, d_H)$ the Hausdorff metric space of compact sets of paths with starting points.

The Brownian web will be a random variable into $(H, \mathcal{F}_H)$, where $\mathcal{F}_H$ is the Borel-$\sigma$-algebra generated by the metric $d_H$. 

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Define \((\mathbb{R}^2, \rho)\) to be the compactification of \(\mathbb{R}^2\) under the metric \(\rho\), given by

\[
\rho((x_1, t_1), (x_2, t_2)) = \frac{\tanh(x_1)}{1 + |t_1|} - \frac{\tanh(x_2)}{1 + |t_2|} \vee |\tanh(t_1) - \tanh(t_2)|
\]

This is clearly a metric (noting the injectivity of \(\tanh(t)\)). Think of \(\mathbb{R}^2\) as the image of \([-\infty, \infty] \times [-\infty, \infty]\) under the mapping \((\Phi(x, t), \Psi(t)) = (\frac{\tanh(x)}{1+|t|}, \tanh(t))\); this is easily seen to yield the compactification of \((\mathbb{R}^2, \rho)\). (Note that \((\Phi(x, t), \Psi(t))\) is by no means the only mapping that achieves that compactification).

Note that \(\mathbb{R}^2\) is also completion, as the only Cauchy sequences in \((\mathbb{R}^2, \rho)\) which fail to have a limit are those in which at least one coordinate tends to infinity or minus infinity. Taking \(\tanh\) in both coordinates maps \([-\infty, \infty] \times [-\infty, \infty]\) onto \([-1, 1] \times [-1, 1]\), then taking \(\frac{1}{1+|t|}\) squeezes the horizontal coordinate even further, depending on the value of \(t\), so as to identify all space points at time \(t = -\infty\) to a single point and those at time \(t = \infty\) to another single point. So one has created a suitable compactification of \(\mathbb{R}^2\) on which the spaces \(\Pi, H\) are complete and separable to allow for weak convergence results concerning the Brownian web.

Secondly we define a path space: for any \(t_0 \in [-\infty, \infty]\), let \(C[t_0]\) denote the set of functions \(f\) from \([t_0, \infty]\) into \([-\infty, \infty]\) such that \(\Phi(f(t), t)\) is continuous (so the paths starting at \(t_0\) are continuous with respect to the topology on \(\mathbb{R}^2\)).

Define

\[
\Pi = \bigcup_{t_0 \in [-\infty, \infty]} C[t_0] \times \{t_0\}.
\]

The most ‘tricky’ step is to put a metric on this as the paths may have different starting points. Let \((f, t_0) \in \Pi\) – a path in \(\mathbb{R}^2\) starting at \((f(t_0), t_0)\). Denote by \(\hat{f}\) the function that extends \(f\) to all of \([-\infty, \infty]\) by setting it equal to \(f(t_0)\) for \(t < t_0\).

Define a distance on \(\Pi\) by

\[
d((f_1, t_1), (f_2, t_2)) = (\sup_t |\Phi(\hat{f}_1, t) - \Phi(\hat{f}_2, t)|) \vee |\Psi(t_1) - \Psi(t_2)|,
\]

to obtain a suitable sup-metric over the extended paths and their starting points. By the choice of metric on the underlying space \(\mathbb{R}^2\), note that \(\Pi\) is bounded.

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Finally, let $H$ denote the set of compact subsets of $(\Pi, d)$ with $d_H$ the induced Hausdorff metric

$$d_H(K_1, K_2) = \sup_{g_1 \in K_1} \inf_{g_2 \in K_2} d(g_1, g_2) \vee \sup_{g_2 \in K_2} \inf_{g_1 \in K_1} d(g_1, g_2)$$

Note that $d_H(K_1, K_2) \in \mathbb{R}$, as the underlying space $\Pi$ is bounded, and moreover equivalent to

$$d_H(K_1, K_2) := \inf\{r \in [0, \infty) \mid K_1 \subseteq (K_2)_r \text{ and } K_2 \subseteq (K_1)_r\}$$

where $S_r$ denotes $B_r(S) = \{x \in E : d(x, s) < \epsilon \text{ for some } s \in S\}$. This is the usual Hausdorff metric space on compact subsets of a metric space.

We verify the following claim, which is not proved in the literature.

**Proposition 8** $(\Pi, d)$, $(H, d_H)$ are complete separable metric spaces.

**Proof:** To verify the axioms note that positivity, symmetry and homogeneity are obvious from the definition. In the case of $(\Pi, d)$ non-degeneracy follows essentially from the injectivity of tanh and the triangle inequality is as in the usual sup-metric case. For $(H, d_H)$ check that $d_H(K_1, K_2) = 0 \Rightarrow K_1 = K_2$ as follows: Note that $K_1 \subseteq (K_2)_\epsilon$ for any $\epsilon > 0$. Also $\bigcap_{\epsilon}(K_2)_\epsilon = K_2$, since $K_2$ is closed. Hence $K_1 \subseteq \bigcap_{\epsilon}(K_2)_\epsilon = K_2$, by symmetry $K_1 = K_2$. The triangle inequality is also easily checked: if for $K, L, M$ in $H$, $d_H(K, L) = r$, $d_H(L, M) = s$, then $K \subseteq M_{r+s}$, $M \subseteq K_{r+s}$, by the triangle inequality for $d$ on $\Pi$, so $d_H(K, M) \leq r + s$.

Next let us prove the completeness of $\Pi$. Let $(f_n, t_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\Pi$, $\epsilon > 0$, so there exists $N$ such that for $m, n \geq N$, $d((f_n, t_n), (f_m, t_m)) < \epsilon$. By completeness of $\mathbb{R}^2$, continuity of $\Phi, \Psi$ we have $t_n \rightarrow t_0 \in [-\infty, \infty]$ and a pointwise limit $(f(t + t_0), t_0)$ ($t \geq 0$) obtained by letting $(f_n(t + t_n), t_n) \rightarrow (f(t + t_0), t_0))$ as $n \rightarrow \infty$. Then, for any $t$

$$\sup_{t} |\Phi(\hat{f}_n(t), t) - \Phi(\hat{f}_m(t), t)| < \epsilon,$$

so letting $m \rightarrow \infty$ implies uniform convergence of $f_n$ to $f$, so $d((f_n, t_n), (f, t_0)) \rightarrow 0$ as $n \rightarrow \infty$. $f$ is in $C[t_0] \subset \Pi$ as by uniform convergence it is continuous, even with respect to $p$. So $(\Pi, d)$ is complete.

Completeness of $H$ is, for an underlying complete metric space, a standard result, but not straightforward, so we will not prove it, but refer to [27, p. 171 ff.].

Now for the separability of $\Pi$: first we prove that $C[q_0] \times \{q_0\}$ for $q_0 \in \mathbb{Q}$ is separable. Let $\epsilon > 0$ and $K$ so large that for $|t| \geq K$

$$\sup_{|t| \geq K} |\Phi(\hat{f}_1, t) - \Phi(\hat{f}_2, t)| < \frac{\epsilon}{2}$$

for any $f_1, f_2 \in C[q_0] \times \{q_0\}$. This is possible by definition of $\Phi$. From the Weierstrass Approximation Theorem one obtains a countable dense set $P_{q_0}$ of functions (polynomials with rational coefficients) in $C([-K, K])$ with the Euclidean metric on $[-K, K]$.

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However then this set is also dense with respect to the metric $d$, essentially because \( \tanh \) is a contraction.

Thus $C[q_0] \times \{q_0\}$ is separable, as we have that for any $f$ in that space there exists $g \in P_{q_0}$ such that

\[
\sup_{t \in [-K,K]} |\Phi(\hat{f}, t) - \Phi(\hat{g}, t)| < \epsilon, \frac{\epsilon}{2},
\]

and therefore

\[
\sup_{t} |\Phi(\hat{f}, t) - \Phi(\hat{g}, t)| < \epsilon,
\]

as outside the interval $[-K,K]$ the distance is smaller than $\frac{\epsilon}{2}$ anyway.

Finally for any $t_0$ in $\mathbb{R}$, take $q_0 \in \mathbb{Q}$ with $q_0 < t_0$. If $f \in C[t_0] \times \{t_0\}$, then there exists a continuous $g \in C[q_0] \times \{q_0\}$ with $g = f$ on $[t_0, \infty)$ and $g = f(t_0)$ on $[q_0, t_0]$ so that by the choice of metric on $\Pi$, there is a $p \in P_{q_0}$ such that $d(p, g) = d(p, f) < \epsilon$. Thus $(\Pi, d)$ is separable.

The separability of the Hausdorff metric space built on a separable metric space $(E, d)$ is a standard result. Unable to find a reference, we give a proof after an idea of R. Sun [pers. comm.]: let $D$ be the Hausdorff metric space built on $E$, so that $E$ is the closure of $D$ and $D$ is dense in $E$. Let $(H, d_H)$ be the Hausdorff metric space built on $E$, so that it is sufficient to exhibit a countable set $Q \subseteq H$ that is dense in $H$. Choose $Q$ to be the set of all finite subsets of $D$, then $Q$ is countable. Let $\epsilon > 0$ and $K \in H$. As $D$ is dense in $E$, the $\epsilon$-balls of all points in $D$ form an open cover of $E$, so of $K$. So $K$ has a finite subcover: the $\epsilon$-balls around finitely many points $d_1, \ldots, d_n \in D$. $\{d_1, \ldots, d_n\} = K_2 \subseteq Q$ and so $d_H(K, K_2) < \epsilon$ by definition of $d_H$. $\square$

### 2.2 Construction of the Brownian web

A space for the Brownian web being introduced, we can construct the object and prove the following existence and uniqueness theorem.

**Theorem 9 (Theorem 2.1. of [12])** There exists an $(H, \mathcal{F}_H)$-valued random variable $\overline{W}$ whose distribution is uniquely determined by three properties:

1. From any deterministic point $(x, t) \in \mathbb{R}^2$ there is a.s. a unique path $W_{x,t}$ starting from $(x, t)$.

2. For any deterministic $n$ and $(x_1, t_1), \ldots, (x_n, t_n)$ the joint distribution of $(W_{x_1, t_1}, \ldots, W_{x_n, t_n})$ is that of coalescing Brownian motions.

3. For any deterministic dense countable subset $D$ of $\mathbb{R}^2$ a.s. $\overline{W}$ is the closure of $\{W_{x,t} : (x,t) \in D\}$ taken in $(\Pi, d)$, so that $\overline{W}$ maps into $(H, d_H)$.

**Remarks:** Property (3) is to be understood as saying that the closures of $\{W_{x,t} : (x,t) \in D\}$ for any $D$ are the same object in distribution. Regarding property (2) we mean by the finite-dimensional distributions of $\overline{W}$ the induced probability measures $\mu(x_1, t_1), \ldots, (x_n, t_n)$ on the subset of $\Pi$ of paths starting from the finite deterministic set of those points $(x_1, t_1), \ldots, (x_n, t_n)$ in $\mathbb{R}^2$. We note that property (3) is essential to
picking out an object uniquely in distribution, as we shall prove. This is precisely where there is a choice what to assume on the Brownian web beyond the intuitively required properties (1) and (2) and where the work of Fontes et al. [12] extends Arratia’s work we outlined in Section 1.3. Whereas properties (1) and (2) yield that \(W\) contains the paths required to form the Brownian web, (3) ensures that there are not too many extraneous paths.

The proof of the theorem has several steps. First we construct the coalescing Brownian motions on a countable dense set and define \(W\) (Definition 10), then we prove in Lemmas 11 - 14 that \(W\) satisfies the conditions in Theorem 9.

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space where an i.i.d. family of Brownian motions \((B^j)_{j \in \mathbb{N}}\) is defined. Fix \(D = \{(x_j, t_j), j \geq 1\}\), a countable dense set in \(\mathbb{R}^2\). Let \(W^j\) be a Brownian path starting at site \(x_j\), time \(t_j\), that is

\[
W^j(t) = x_j + B^j(t - t_j) \quad t \geq t_j.
\]

As briefly discussed in section 2.2., one can now form \(N\) coalescing Brownian motions by applying a collision rule to \(W^1, ..., W^N\), denoting the resulting paths by \(\tilde{W}^j\) for \(1 \leq j \leq N\); and we know that any collision rule will produce the same object in distribution. To be precise, Fontes et al. (in [12]) choose to set \(\tilde{W}^1 = W^1\) and for \(j \geq 2\), \(\tilde{W}^j\) to be a path from \([t_j, \infty)\) into \(\mathbb{R}^2\) by recursion as follows. Set

\[
\tau_j = \inf\{t \geq t_j : W_j(t) = \tilde{W}_i(t) \text{ for } 1 \leq i \leq j - 1\}
\]

\[
I_j = \min\{1 \leq i \leq j - 1 : W_j(\tau_j) = \tilde{W}_i(\tau_j)\}.
\]

Exploiting the continuity of paths and using the identity permutation for the collision rule define:

\[
\tilde{W}^j(t) = \begin{cases} 
W^j(t) & \text{if } t_j \leq t \leq \tau_j \\
W^{I_j}(t) & \text{if } t > \tau_j
\end{cases}
\]

Then define the Brownian web skeleton \(W(D)\), starting from the set \(D\), by

\[
W^n = W^n(D) = \{\tilde{W}^j, 1 \leq j \leq n\}
\]

\[
W = W(D) = \bigcup_{n \in \mathbb{N}} W^n
\]

**Definition 10 (Brownian web, contemporary definition, after [12])** The Brownian web \(W(D)\) is the closure of \(W(D)\), the Brownian web skeleton, taken in \((\Pi, d)\).

Note that, by the choice of metric \(d\), it is immediate that in the Brownian web, (at least) one path starts from every space-time point. The proof of Theorem 9 is presented as a combination of the following lemmas.

**Lemma 11 (Proposition 3.2. of [12])** \(W(D)\) is almost surely a compact subset of \((\Pi, d)\), so \(W(D) \in H\) a.s.
Lemma 12 (Proposition 3.1. of [12]) \( \overline{W}(D) \) satisfies (1), (2) of Theorem 9: its finite-dimensional joint distributions (whether the paths start from points in \( D \) or not) are those of coalescing Brownian motions, and from every point in \( \mathbb{R}^2 \) the path starting there is a.s. unique.

Lemma 13 (Proposition 3.5. of [12]) The distribution of \( \overline{W}(D) \) does not depend on \( D \) nor on its order (so one can construct the Brownian web from any dense countable subset) and so \( \overline{W}(D) \) satisfies property (3) of the above theorem.

Lemma 14 The distribution of \( \overline{W}(D) \) is uniquely determined by (1), (2), (3) in Theorem 9.

Note that these lemmas trivially imply Theorem 9.

Remark: It is also necessary to verify that \( \overline{W} \) is actually a mapping measurable with respect to \( \mathcal{F}_H \) - this is done in an appendix of [12], which we will omit because it is a technical proof involving to show that \( \mathcal{F}_H \) is generated by certain 'cylinders' in \( H \).

Proof of Lemma 11: Recall the definition of tightness (cf. Section 1.2). We deduce from the following technical result (proved in [12, p. 43 - 47]) lemma 11.

Lemma 15 (Proposition B5, [12]) Let \( D \) be a countable dense subset of \( \mathbb{R}^2 \), \( \mu_k \) the distribution of the \((H, \mathcal{F}_H)\)-valued random variable \( W_k = W_k(D) = \{ \tilde{W}_1, \ldots, \tilde{W}_k \} \). Then the family of measures \( \{ \mu_k : k \in \mathbb{N} \} \) is tight.

Lemma 11 is then established via the somewhat more general

Claim: If \( W_n \) is an a.s. increasing sequence of \((H, d_H)\)-valued random variables and the family of distributions \( \{ \mu_n : n \in \mathbb{N} \} \) of \( W_n \) is tight, then \( \bigcup_n W_n \) is a.s. compact in \((\Pi, d)\).

Proof of claim: Let \( V_k \) be an increasing sequence of points in \((H, d_H)\) converging in the \( d_H \) metric to some \( V \in H \). As the \( V_k \) are increasing, we have by the properties of \( d_H \) that \( V_k \subseteq V \) for all \( k \). So \( \bigcap_k V_k \subseteq V \) and hence \( \bigcup_k V_k \) is compact as a closed subset of a compact set \( V \).

From the definition of tightness, given \( \epsilon > 0 \), there exists compact \( K \subseteq H \), such that \( \mu_n(K) = \mathbb{P}(W^{-1}_n(K)) \geq 1 - \epsilon \) for all \( n \), so by downward monotonicity \( \mathbb{P}(W^{-1}_n(K)) \geq 1 - \epsilon \). This is because \( \bigcap_n \{ W_n \in K \} = \{ W_m \in K \} \) so that \( \{ W_m \in K \} \to \bigcap_n \{ W_n \in K \} \) as \( m \to \infty \).

If \( W_n \in K \) for all \( n \), \( K \) compact, there is a subsequence \( W_{n_j} \) which converges to a point in \( K \), thus converges in \( H \). So \( \bigcup_j W_{n_j} (= \bigcup_n W_n) \) is a compact subset, by the first part of the proof. So \( \mathbb{P}(\bigcup_n W_n) \) is a compact subset of \( \Pi \) \( \geq 1 - \epsilon \). \( \epsilon \) was arbitrary, so the claim follows.

Proof of Lemma 12: This is a ‘trapping’ argument and the key to the whole construction as it exploits intrinsic properties of Brownian motion and the setting of the path space \((\Pi, d)\). We will prove that for any single deterministic point \((x, t)\) in \( \mathbb{R}^2 \) there is a.s. a unique path distributed as (coalescing) Brownian motion starting from it. As in the proof we only use ‘local’ properties and paths near the single point, it
is clear that the statement will hold for finitely many deterministic points. Moreover
we can w.l.o.g. assume that \((x, t) = (0, 0)\) to simplify notation.

The result will follow from combining several estimates: Let \(\{(x_n^r, t_n^r)\}_{n=1}^{\infty}\),
\(\{(x_n^l, t_n^l)\}_{n=1}^{\infty}\) be two sequences of points of \(D\) satisfying:

There exist constants \(0 < c_1, c_2 \leq \frac{1}{2}\) such that

\[-\frac{c_1}{n^2} < x_n^l < 0 < x_n^r < \frac{c_2}{n^2}\] and \((x_n^l)^3 < t_n^l < 0\); \((-x_n^r)^3 < t_n^r < 0\).

Let \(\hat{W}_{n,l}, \hat{W}_{n,r}\) be coalescing Brownian motions starting from \((x_n^l, t_n^l), (x_n^r, t_n^r)\) respectively, \(B\) a standard Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\).

Recall the reflection principle (see for instance [25, p. 105]):

For \(a > 0\), \(t \geq 0\), \(\mathbb{P}\{S_t \geq a\} = 2\mathbb{P}\{|B(t)| \geq a\} = \mathbb{P}\{|B(t)| \geq a\}\) where
\(S_t = \sup_{s \leq t} B(s)\).

We calculate

\[\mathbb{P}\{\max_{t_n^l \leq s \leq 0} \hat{W}_{n,l}(s) \geq 0\} = 2\mathbb{P}\{\hat{W}_{n,l}(0) \geq 0\} = 2\mathbb{P}\{|B(t_n^l)| \geq |x_n^l|\},\]

by the reflection principle.

Then, by scaling, this equals

\[2\mathbb{P}\{B(1) \geq \frac{|x_n^l|}{|t_n^l|}\} \leq 2\mathbb{P}\{B(1) \geq \sqrt{2}n\} \leq e^{-n^2}\] (2.1)

using the conditions on \(c_1, c_2, x_n^l, t_n^l, x_n^r, t_n^r\) above and estimating the integral to which
the second last term is equal.

Let \(\tau_n = \max\{t_n^l, t_n^r\}\). By performing similar calculations, using the reflection
principle and the scaling property of Brownian motion, one obtains the following
further estimates (for the details see [12, p.10-12]):

\[\mathbb{P}\{\min_{t_n^l \leq s \leq 0} \hat{W}_{n,r}(s) \leq 0\} \leq e^{-n^2}\] (2.2)

\[\mathbb{P}\{\inf_{\tau_n \leq s \leq \frac{1}{n}} |\hat{W}_{n,l}(s) - \hat{W}_{n,r}(s)| > 0\} \leq \frac{C}{n^2}\] (2.3)

for some positive constant \(C\);

\[\mathbb{P}\{\max_{t_n^l \leq s \leq \frac{1}{n}} \hat{W}_{n,l}(s) \geq n^{-\frac{1}{4}}\} \leq e^{-C' n^{\frac{1}{2}}}\] (2.4)

\[\mathbb{P}\{\inf_{t_n^l \leq s \leq \frac{1}{n}} \hat{W}_{n,l}(s) \leq -n^{-\frac{1}{4}}\} \leq e^{-C' n^{\frac{1}{2}}}\] (2.5)

for some positive \(C'\).

Set \(T_n = \inf\{s : \hat{W}_{n,l}(s) = \hat{W}_{n,r}(s)\}\) and \(M_n = \hat{W}_{n,l}(T_n)\). Recall that the first
Borel-Cantelli lemma states that for a sequence of events \((A_n)_{n \in \mathbb{N}}\), if \(\sum_k \mathbb{P}\{A_k\} < \infty\)

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then $\mathbb{P}(\limsup A_k) = 0$ ($\iff \mathbb{P}(\liminf A_k^c) = 1$). So we can conclude from the above using the integral test to estimate the infinite series in the cases (2.1), (2.2), (2.4), (2.5) that a.s. for large $n \in \mathbb{N}$

$$
\tilde{W}_{n,t}(0) < 0 \quad \text{from (2.1)}
$$

$$
\tilde{W}_{n,r}(0) > 0 \quad \text{from (2.2)}
$$

$$
T_n < \frac{1}{n} \quad \text{from (2.3)}
$$

$$
|M_n| < n^{-\frac{1}{2}} \quad \text{from (2.4) and (2.5)}
$$

Let $\Delta_n$ be a ‘triangular’ region of $\mathbb{R}^2$ with vertices $(x_n^1, t_n^1)$, $(x_n^2, t_n^2)$, $(M_n, T_n)$, base $[x_n^1, x_n^2]$ and ‘sides’ $\tilde{W}_{n,l}$, $\tilde{W}_{n,r}$. From the estimates obtained via the Borel-Cantelli lemma, we have that there exists an a.s. finite random variable $N$ such that for all $k > N$ and any coalescing Brownian motion $\tilde{W}'_i$ starting from the point $(x_i, t_i) \in D \cup \Delta_k$

$$
\tilde{W}'_i(s) = \tilde{W}_{k,l}(s) = \tilde{W}_{k,r}(s) \quad \text{for all } s > T_k. \quad (2.6)
$$

Note that $(T_n, M_n) \to 0$ a.s., observe also that from (2.1), (2.2), eventually $\Delta_n$ contains the origin, as we were estimating over the supremum of the paths. So for any sequence $(x_i, t_i) \in D$ converging to $(0, 0)$ we have that the coalescing Brownian motions starting from $(x_i, t_i)$ converge (in the path space metric $d$) to a unique coalescing Brownian motion, starting from $(0, 0)$.

By (2.6) this is independent of the sequence $(x_i, t_i)$ chosen, so a.s. there is a unique Brownian motion starting from $(0, 0)$.

Proof of Lemma 13: Consider two countable dense sets in $\mathbb{R}^2$, $D_1$, $D_2$; what follows is a kind of coupling of the Brownian web constructed from $D_1$ with paths starting on $D_2$. Let $W_k(D_2)$ be as in the construction of the Brownian web, starting with $D_2$. Let $W'_k(D_2)$ be the paths of $\overline{W}(D_1)$ starting from the first $k$ elements of $D_2$. By the previous lemma, $W'_k(D_2)$ has the same distribution as $W_k(D_2)$, that of coalescing Brownian motions. It follows from the fact the $W_k$ and $W'_k$ converge to the respective closures a.s. as $k \to \infty$ and that their a.s. convergence (in $H$) implies convergence in distribution that

$$
\overline{W}(D_2) \sim \overline{W}'(D_2) = \bigcup_k W'_k(D_2) \subseteq \overline{W}(D_1).
$$

where $\sim$ denotes equality in distribution.

Next we prove

$$
\overline{W}(D_1) \subseteq \overline{W}'(D_2).
$$

Let $W_{(x_1, t_1)}$ be a path of $\overline{W}(D_1)$ starting at $(x_1, t_1)$ in $D_1$. Let $(x^3_j, t^3_j)$ be a sequence of points in $D_2$ converging to $(x_1, t_1)$. Recall that by property (1) in Theorem 9 both in $\overline{W}(D_1)$, $\overline{W}'(D_2)$ from every point there is a.s. a unique path starting from it. The paths $W'_{(x^3_j, t^3_j)}$ in $\overline{W}(D_1)$ converge to $W'_{(x_1, t_1)}$. This follows as $(x^3_j, t^3_j)$ tend to $(x^1, t^1)$ using the argument that proves the previous lemma.

Then it follows that $W_{(x_1, t_1)} \in \overline{W}'(D_2)$ because $W_{(x^3_j, t^3_j)} \in W'_k(D_2)$ for $k$ sufficiently large. Finally, from the definition of $\overline{W}(D_1)$ as the closure of paths starting from $D_1$
2.3 Characterising the Brownian web using counting random variables

in the path space metric, it follows that all paths in \( W(D_1) \) can be approximated by paths in \( W_k(D_2) \) and hence \( W(D_1) \subseteq W_k(D_2) \). Combining the two relations we get that

\[
W(D_1) = W'(D_2) \sim W(D_2).
\]

This establishes the distributional independence of the countable dense set, and shows that \( W \) satisfies property (3) of theorem 9: for any countable dense set \( D \), \( W \) is the closure of \( \{ W_{x,t} : (x,t) \in D \} \) in the sense that the constructed object has for any \( D \) the same distribution.

Proof of Lemma 14: Let \( V \) and \( W \) be two \((H,F_H)\)-valued random variables satisfying both (1), (2), (3) of Theorem (9). We show that \( V \) and \( W \) have the same distribution. Fix a dense countable subset \( D \). Let \( W_k(D) \), the version of the Brownian web from \( D \) constructed above, but of which the distribution does not depend on \( D \) as shown by the previous lemma. Using (3) on \( V \), \( V \sim V(D) \), a version of \( V \) constructed from \( D \) as well. Then using (1), (2) on \( W(D) \) and \( V(D) \), we have that for \( \{(x_1,t_1),..., (x_n,t_n)\} \in D, W_{x_1,t_1},..., W_{x_n,t_n} \) and \( V_{x_1,t_1},..., V_{x_n,t_n} \) have the same distribution, namely that of finitely many coalescing Brownian motions. As in the previous lemma, we note \( W_k(D) \to W(D) \) and \( V_k(D) \to V(D) \) as \( k \to \infty \) a.s. in the space \((H,d_H)\). Hence also the distributions of \( W_k(D) \) converge to that of \( W(D) \) and those of \( V_k(D) \) to that of \( V(D) \). So \( W \sim W(D) \sim V(D) \sim V \) and the uniqueness in distribution is established. This finishes the proof of Theorem 9. \( \Box \)

2.3 Characterising the Brownian web using counting random variables

To be able to consider convergence to the Brownian web (in Chapter 3), we need to give an equivalent characterisation of the Brownian web in terms of variables counting paths. We state the main results and some crucial arguments only.

Definition 16 Let \( t > 0, t_0, a, b \in \mathbb{R} \) with \( a < b \). Then let \( \eta_W(t_0, t, a, b) \) be the number of distinct points in \( \mathbb{R} \times \{ t_0 + t \} \) that are touched by paths in \( W \) which also touch some point in \( [a,b] \times \{ t_0 + t \} \).

So \( \eta_W \) counts how many Brownian web paths through \([a,b]\) at \( t_0 \) have ‘survived’ at \( t_0 + t \). The definition will be applied to other collections of paths analogously. For the purpose of arriving at a convergence result in Chapter 3 we will need the following alternative characterisation theorem:

Theorem 17 (Theorem 4.6. of [12]) Let \( W' \) be an \((H,F_H)\)-valued random variable, \( D \) a deterministic countable dense subset of \( \mathbb{R}^2 \), and for each \( y \in D \), let \( \theta^y \in W' \) be some single random path starting at \( y \). Then \( W' \) has the distribution of the Brownian web \( W \) if

(1) the \( \theta^y \) are distributed as coalescing Brownian motions.
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(2) for all \( t_0, t, a, b \) \( \mathbb{P}(\eta_{W^*} \geq k) \leq \mathbb{P}(\eta_W \geq k) \) for all \( k \).

We will not prove this (but cf. [12, section 5]); instead we give some ideas that go into the proof:

**Definition 18**: (stochastic ordering on the Hausdorff metric space \( H \)) Let \( g \) be a bounded measurable function \( g : (H, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}) \) that is increasing (\( K \subseteq K' \) implies \( g(K) \leq g(K') \)): If \( \mu_1, \mu_2 \) are measures on \( H \) and \( \int g d\mu_1 \leq \int g d\mu_2 \) for all \( g \), write \( \mu_1 \ll \mu_2 \).

**Remark**: \( \ll \) defines a weak partial order on the measures on \( H \), where antisymmetry follows (with work) by representing indicator functions on \( H \) as the difference of bounded increasing functions.

We show that constructing the Brownian web as a closure from a countable dense subset is equivalent to another minimality property:

**Proposition 19**: (Theorem 3.6. of [12]) In Theorem 9 above, one can replace condition (3) by the following condition:

\((3')\) If \( W^* \) is any other \((H, \mathcal{F}_H)\)-valued random variable satisfying (1), (2) in Theorem 9, then \( \mu_W \ll \mu_{W^*} \).

So there exists an \((H, \mathcal{F}_H)\)-valued random variable \( W \) whose distribution is uniquely determined by (1), (2), (3').

**Proof**: Existence: We prove that the constructed Brownian web \( W \) satisfies condition (3'). Let \( W^* \) satisfy (1), (2). By (3), we have independence of distribution of the underlying countable dense set for \( W \). So choose \( D \) a deterministic dense countable subset and construct \( W \) started from \( D \) as the closure of paths of \( W^* \) started from \( D \). So \( W \subseteq W^* \) a.s., and by assumption on \( g \),

\[
\int g \, d\mu_W \leq \int g \, d\mu_{W^*}
\]

so \( \mu_W \ll \mu_{W^*} \). For the uniqueness argue as follows: if \( W, W^* \) have different distributions and both satisfy (1), (2), (3') above, then \( \mu_W \ll \mu_{W^*}, \mu_{W^*} \ll \mu_W \) so that \( \int g \, d\mu_W = \int g \, d\mu_{W^*} \) for all \( g \) bounded, increasing. So by the above remark, \( \mu_W = \mu_{W^*} \).

Next we quote without proof the following lemma, that we will need later:

**Lemma 20**: (Proposition 4.1. of [12]) For the Brownian web skeleton the counting random variable \( \eta_D = \eta_{W(D)}(t_0, t, a, b) \) is a.s. finite with finite expectation, and

\[
\mathbb{P}(\eta_D \geq k) \leq \mathbb{P}(\eta_D \geq k - 1)\mathbb{P}(\eta_D \geq 1).
\]
2.3 Characterising the Brownian web using counting random variables

Remark: The proof of this relies on an approximation of the Brownian motions by discrete time simple symmetric coalescing random walks. The essential ingredient is an FKG (‘Fortuin-Kasteleyn-Ginibre’) inequality; it is beyond scope of this dissertation to go into the details. We refer to [12, p.14-15] for the proof and to [16, ch. 2.1.-2.3] for a discussion of FKG-type correlations. However Lemma 20 enables us to prove the following:

**Proposition 21 (Proposition 4.3. of [12])** Almost surely for every $\epsilon > 0$, every path $\theta = (f(s), t_0)$ in $\overline{W}(D)$, there exists a path $\theta_\epsilon = (g(s), t_0')$ in the skeleton $W(D)$ such that $g(s) = f(s)$ for all $s \geq t_0 + \epsilon$.

**Proof:** Let $\epsilon > 0$ be given. Since $\overline{W}$ is the closure in $(\Pi, d)$ of $W(D)$ we know there is a sequence $\theta_n = \{(g_n(s), t_n')\}$ of paths in $W(D)$ such that $d(\theta_n, \theta) \to 0$ as $n \to \infty$. So there exists $M$ large such that for $n \geq M$, $t_n' < t_0 + \frac{\epsilon}{2}$. Define an integer valued random variable $N$ by $f(t_0 + \frac{\epsilon}{2}) \in [N, N + 1]$.

From the above lemma we have that $\eta_D(t_0 + \frac{\epsilon}{2}, N - 1, N + 2)$ is a.s. finite. $\theta_n \to \theta$ in $d$ implies that $g_n(t_0 + \frac{\epsilon}{2}) \in [N - 1, N + 2]$ eventually. Then $g_n(t_0 + \epsilon)$ is eventually constant a.s. because otherwise $\eta_D(t_0 + \frac{\epsilon}{2}, N - 1, N + 2)$ would not be finite. But as $d(\theta_n, \theta) \to 0$ this implies $\theta(t_0 + \epsilon) = \theta_n(t_0 + \epsilon)$ for large enough $n$, so the coalescing property implies the proposition. $\square$

Note that this is closely related to Theorem 7. One can prove Theorem 17 by deriving a result similar to Lemma 20 for $\eta_{\Pi(D)}$, the counting variable of $W$ and comparing it to that of $W'$ ($W'$ as defined in Theorem 17).

In Appendix B we discuss further very important theory of the Brownian web.

The construction and characterisation of the Brownian web accomplished, we have built the foundation for results about the convergence to the Brownian web, which we will now consider.
Chapter 3

Convergence to the Brownian web

In this chapter we deal with results concerning the convergence to the Brownian web, as presented in [8], [12], [30]. The main motivation for studying the Brownian web is that one intuitively expects it to be the scaling limit of simple configurations of random paths in the plane. In the following we discuss how weak convergence results for the Brownian web in the setting introduced in the previous chapter can be proved. After presenting a general convergence theorem, we deal with the convergence to the Brownian web of simple coalescing random walks, non-simple coalescing random walks and a configuration of paths called “Poisson tree”.

3.1 The convergence theorem of [12]

Fontes et al. prove in [12] a general convergence theorem for the Brownian web concerning weak convergence of measures on $H$ to the Brownian web distribution. It is a more subtle result than Donsker’s Invariance Principle both because the Brownian web is a far more complicated limiting object compared to Brownian motion and because the theorem of Fontes et al. can be applied to any set of paths, not merely simple random walks. Before stating sufficient conditions for convergence to the Brownian web, we need some more definitions and notation:

Let $a, b, t_0 \in \mathbb{R}$, $t > 0$, $K \in H$. Define two real-valued functions on $H$ as follows

$$l_{t_0,t}([a,b])(K) = \inf \{x \in [a,b] \mid \exists y \in \mathbb{R} \text{ and a path in } K \text{ which touches both } (x, t_0) \text{ and } (y, t_0 + t)\}$$

$$r_{t_0,t}([a,b])(K) = \sup \{x \in [a,b] \mid \exists y \in \mathbb{R} \text{ and a path in } K \text{ which touches both } (x, t_0) \text{ and } (y, t_0 + t)\}$$

Then define the following functions on $H$ taking subsets of $\mathbb{R}$ as values:

$$N_{t_0,t}([a,b])(K) = \{y \in \mathbb{R} \mid \exists x \in [a,b] \text{ and a path in } K \text{ which touches both } (x, t_0) \text{ and } (y, t_0 + t)\}$$

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\[ N_{t_0,t}^\pm(\{a,b\})(K) = \{ y \in \mathbb{R} \mid \text{there is a path in } K \] which touches both \((l_{t_0,t}(\{a,b\})(K),t_0)\) and \((y,t_0+t)\) \]

\[ N_{t_0,t}^\pm(\{a,b\})(K) = \{ y \in \mathbb{R} \mid \text{there is a path in } K \] which touches both \((r_{t_0,t}(\{a,b\})(K),t_0)\) and \((y,t_0+t)\) \]

Note that \(|N_{t_0,t}(\{a,b\})(K)| = \eta_K(t_0,t,a,b) (\eta_K as defined in Section 2.3); note because \(K\) is compact \(N_{t_0,t}^+(\{a,b\})(K), N_{t_0,t}^-(\{a,b\})(K)\) are well-defined. We will in the following suppress the variable \(K\) for readability of the formulæ.

**Definition 22 (convergence conditions)** \ Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of \((H, \mathcal{F}_H)\)-valued random variables with distributions \((\mu_n)_{n \in \mathbb{N}}\) on \(H\). Define three conditions as follows:

(I) there exists path-valued random variables \(\theta_n \in X_n \ (y \in \mathbb{R}^2)\) satisfying: for \(D\) a deterministic countable subset in \(\mathbb{R}^2\), deterministic \(y_1, ..., y_n \in D\), \(\theta_{y_1}^n, ..., \theta_{y_n}^n\) converge jointly in distribution as \(n \to +\infty\) to coalescing Brownian motions starting at \(y_1, ..., y_m\).

\[(B_1) \ \forall \beta > 0 \ \limsup_{n \to \infty} \sup_{t > \beta} \sup_{a \in \mathbb{R}} \mu_n(|N_{t_0,t}(\{a-\epsilon,a+\epsilon\})| > 1) \to 0 \text{ as } \epsilon \to 0^+\]

\[(B_2) \ \forall \beta > 0 \ \frac{1}{\epsilon} \limsup_{n \to \infty} \sup_{t > \beta} \sup_{a \in \mathbb{R}} \mu_n(N_{t_0,t}(\{a-\epsilon,a+\epsilon\}) \neq (N_{t_0,t}^+(\{a-\epsilon,a+\epsilon\}) \cup (N_{t_0,t}^-(\{a-\epsilon,a+\epsilon\})) \to 0 \text{ as } \epsilon \to 0^+\]

**Remark:** Condition (I) is the expected condition that there must be paths in the sets \(X_m\) that converge to Brownian motions. Note that the \(\theta_{y_1}^n, ..., \theta_{y_n}^n\) are not required to start at \(y_1, ..., y_m\), only their starting points must converge to the \(y_1, ..., y_m\).

Condition \((B_1)\) ensures that for any subsequential limit \(X\) of \((X_n)_{n \in \mathbb{N}}\) with distribution \(\mu_X\) and any \(a \in \mathbb{R}\) there is \(\mu_X\)-a.s. at most one path starting from \(a\).

Condition \((B_2)\) ensures that the rate of such convergence in the case that there are more than the two boundary paths is fast, of order \(o(\epsilon)\) as \(\epsilon \to 0\). So (I) ensures that the right paths are in \((X_n)\) to form the limiting object, whereas \((B_1), (B_2)\) ensure one does not have too many paths.

We can now state the convergence theorem for the Brownian web:

**Theorem 23 (Theorem 6.3. of [12])** \ Suppose that \(\{\mu_n : n \in \mathbb{N}\}\) is tight. If (I), \((B_1), (B_2)\) hold then \((X_n)_{n \in \mathbb{N}}\) converges in distribution to the Brownian web \(\overline{W}\).

Note that this theorem is particularly remarkable as it is not a priori assumed that the collections of paths \((X_n)_{n \in \mathbb{N}}\) are collections of random walks.

**Proof:** The proof is lengthy and involves many manipulations of the above conditions, so instead of recasting the full details here, which can be found in [12, p. 33 - 38], we limit ourselves to explaining some steps and focus on the key argument - our summary of the proof is provided in Section C.1 of Appendix C. \(\Box\)

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3.2 Simple coalescing random walks converge to the Brownian web

In this section we deal with the result that is the analogue of Donsker’s Invariance Principle for the Brownian web: the coalescing simple random walks defined in the introduction converge to the Brownian web in the sense of weak convergence on $H$.

Recall from the introduction that one only considers random walks starting at points $(j, k) \in \mathbb{Z}^2$ with $(j + k)$ even. This is since those starting from ‘odd’ points of the integer lattice evolve independently, they can never interact with random walks on the even starting points. So as a scaling limit one would get two independent copies of the Brownian web. Similarly, if one had that the random walk increment $Y$ takes values $\pm 2$ with equal probability and paths starting from all points in $\mathbb{Z}^2$ there would be four independent copies of the Brownian web in the limit. So we only consider collections of random walks which are irreducible: two random walks from any two starting points have a strictly positive probability of coalescing.

In fact, it is the coalescence property (if $Z_{i_0,j_0}, Z_{i_1,j_1}$ are starting from $(i_0,j_0)$, $(i_1,j_1)$ respectively, then $Z_{i_0,j_0}(n) = Z_{i_1,j_1}(n)$ implies $Z_{i_0,j_0}(m) = Z_{i_1,j_1}(m)$ for all $m \geq n$) that much simplifies verifying convergence.

If one considers a sequence of compact sets of paths $X_n$ (as points of $H$ with distributions $\mu_n$) and the paths are coalescing and non-crossing, then the following monotonicity property holds:

For all $a < b, t_0, 0 < s < t \in \mathbb{R}$

$$\mathbb{P}(|\eta_{X_n(t_0, t, a, b)}| \geq k) \leq \mathbb{P}(|\eta_{X_n(t_0, s, a, b)}| \geq k)$$

for all $n, k \in \mathbb{N}$. It is easy to see that this property fails if coalescence upon coincidence is not assumed, considering the case of just two paths, one starting from $a$, one from $b$ which touch each other but do not coalesce at time $s$, and never meet again afterwards.

It follows from this that in the case of the coalescing simple random walks - because they never cross - the following, simpler, conditions ($B'_1$), ($B'_2$) imply ($B_1$), ($B_2$):

($B'_1$) $\forall t > 0 \limsup_{n \to \infty} \sup_{a,t_0 \in \mathbb{R}^2} \mu_n(\eta(t_0, t, a, a + \epsilon) \geq 2) \to 0$ as $\epsilon \to 0^+$

($B'_2$) $\forall t > 0 \limsup_{n \to \infty} \sup_{a,t_0 \in \mathbb{R}^2} \mu_n(\eta(t_0, t, a, a + \epsilon) \geq 3) \to 0$ as $\epsilon \to 0^+$

The additional supremum taken over $t > \beta$ in ($B_1$) and ($B_2$) vanishes in ($B'_1$) and ($B'_2$) because the above monotonicity holds. Also in ($B_2$) we can replace the $N$-variables by a condition on $\eta$ because in the non-crossing case, if the boundary paths coalesce, then $\eta = 1$ anyway.

Finally we need to incorporate the simple coalescing random walks into the setting of $(H, d_H)$: let $Y$ be the set of random walks from the even starting points in $\mathbb{Z}^2$ with boundary paths added in order to make $Y$ a compact subset of $\Pi$: Add all the paths of the form $(f, s_0)$ with $s_0 \in \mathbb{Z} \cup \{-\infty, +\infty\}$ and $f \equiv \infty$ or $f \equiv -\infty$. (There are two paths starting from $s_0$ iff $s_0 = -\infty$.) Then $Y$ is easily seen to be sequentially compact in $(\Pi, d)$ and hence compact.
3.2 Simple coalescing random walks converge to the Brownian web

Recall that we rescale $Y$ by factors $\frac{1}{n}$ in space and $\frac{1}{n^2}$ in time (see Introduction and Section 1.2) and we shall label the rescaled object $Y^{(n)}$. We have the following result:

**Theorem 24** ("Invariance Principle for the web case", Theorem 7.2. of [12])

The sequence $(Y^{(n)})_{n \in \mathbb{N}}$ of the collections of rescaled simple coalescing random walks (with distribution $\mu_n$) converges in distribution on $H$ to the Brownian web $\bar{W}$ as $n \to \infty$.

Remark: The same result holds for continuous time random walks (see [12, p. 39]).

**Proof:** We prove this, by applying the convergence theorem (Theorem 23). By the preceding discussion it suffices to prove $(B_1^{(n)})$, $(B_2^{(n)})$, $(I)$ and tightness of $\{\mu_n : n \in \mathbb{N}\}$; we limit ourselves to some steps of the proof.

$(I)$ is a consequence of Donsker’s Invariance Principle (Theorem 5). Let $y_1 = (x_1, t_1), \ldots, y_m = (x_m, t_m) \in D$. By Donsker’s Invariance Principle it follows that for any $y_j$, $1 \leq j \leq m$ there exists a random walk $\theta_{n}^{y_j} \in Y^{(n)}$ that converges in distribution to a Brownian motion $B_{y_j}$ started at $x_j$. (Note that the (possibly) different starting space-time points of each Brownian motion and the random walk approximating it do not influence Donsker’s Invariance Principle as $\rho(B_{y_j}(0), \theta_{n}^{y_j}(0)) \to 0$ as $n \to \infty$.)

Then Donsker’s Invariance Principle holds also for $n$ coalescing random walks jointly, as they are independent before collision, so converge in distribution to $n$ coalescing Brownian motions.

Tightness of $\{\mu_m : m \in \mathbb{N}\}$ is longest to verify, so we have to omit it. One can either prove it for the general case of any sequence of sets of paths in which the paths are non-crossing when assuming a weakened version of $(I)$ [12, p. 46-47], or modify the argument to apply it to the simple random walks directly, as in [11].

Finally we establish $(B_1^{(n)})$, $(B_2^{(n)})$ from properties of the random walks. To this end, we quote the following inequality for coalescing, non-crossing random walks following from a version of Lemma 20 for the set of coalescing random walks (so the proof of it involves an FKG-type inequality [12, p. 14-16], essential in verifying $(B_1^{(n)})$ and $(B_2^{(n)})$ at once):

$$\mu_n(\eta(t_0, t, a, a + \epsilon) \geq k) \leq (\mu_n(\eta(t_0, t, a, a + \epsilon) \geq 2))^{k-1} \quad (k \geq 2).$$

Therefore showing $(B_1^{(n)})$ and $(B_2^{(n)})$ reduces to finding an appropriate bound on $\mu_n(\eta(t_0, t, a, a + \epsilon) \geq 2)$. Taking the supremum over $(a, t_0)$ and the lim sup over $n$, this reduces further to finding a bound on the probability that two Brownian motions $B^0$ and $B^1$, started at $(0, \epsilon)$ and $(0, 0)$ respectively, have not coalesced before time $t$. This is because the coalescing random walks are a translation-invariant collection of paths on $\mathbb{R}^2$, the event $\{\eta(t_0, t, a, a + \epsilon) \geq 2\}$ happens if and only if the two boundary paths are not coalesced at time $t$ by the non-crossing property, and the limit of the distribution of the two boundary paths is that of Brownian motion. Hence we use the reflection principle and the fact that the difference of two independent Brownian motions is distributed as Brownian motion run twice as fast. Noting that $B^0 - B^1 + \epsilon$ is a Brownian motion started at 0 with double speed, we can calculate ($\phi$ is the standard
normal distribution function):

\[
P( \sup_{0 \leq s \leq t} B^0_s - B^c_s + \epsilon \geq \epsilon) = 2P(B^0_t - B^c_t + \epsilon \geq \epsilon) = 2(1 - P(B^0_t - B^c_t + \epsilon \leq \epsilon)).
\]

So

\[
1 - P( \sup_{0 \leq s \leq t} B^0_s - B^c_s + \epsilon \geq \epsilon) = 1 - 2 + 2P(B^0_t - B^c_t + \epsilon \leq \epsilon) = 2\phi(\frac{\epsilon}{\sqrt{2t}}) - 1
\]

noting that if \((B^c_t - B^0_t + \epsilon) \sim N_{0, 2t}\), then \(\frac{1}{\sqrt{2t}}(B^c_t - B^0_t + \epsilon) \sim N_{0, 1}\). Finally \(2\phi(\frac{\epsilon}{\sqrt{2t}}) - 1\) is bounded above by \(C\frac{\epsilon}{\sqrt{t}}\) (because \(\frac{1}{\sqrt{2\pi}} \int_{0}^{\frac{\epsilon}{\sqrt{t}}} e^{-\frac{x^2}{2}} dx \leq \frac{\epsilon}{\sqrt{2\pi}}\) is), which for fixed \(t\) tends to 0 as \(\epsilon \to 0\). \(\square\)

We finally remark that it is now straightforward to extend Theorem 24 to the case of coalescing forward and dual coalescing backward random walks (cf. Appendix B) yielding of course the Double Brownian web as the scaling limit [12, p. 39-40].

3.3 The case of non-simple coalescing random walks

That under some additional assumptions even non-simple coalescing random walks converge to the Brownian web is the result of R. Sun’s PhD thesis [30]. The proof of the result amounts to 25 pages, so we can only hint at some ideas.

By a non-simple random walk, we mean an irreducible aperiodic random walk on \(Z\) of which the increment is still assumed to have mean 0 but can now take any integer value and is in particular not symmetric. Paths are allowed to cross before coalescence. Obtain the coalescing non-simple random walks on the integer lattice by letting a random walker start at every point performing a random walk with increment \(Y\). Denote this collection, after the usual rescaling by \((\frac{1}{n}, \frac{1}{n^2})\), by \(X^n\) with distributions \(\mu^n\) for each \(n \in \mathbb{N}\). Sun has proved the following:

**Theorem 25 (after Theorem 1.3.3. of [30])** If \(E(|Y|^5) < +\infty\) then \(X^n\) converges to the Brownian web in distribution on \(H\) as \(n \to \infty\).

Naturally Sun’s proof is based on the general convergence theorem (Theorem 23 above) of Fontes et al.. It does not contain a verification of condition \((B_2)\) of the convergence theorem for the non-simple case, instead one uses a duality argument to reformulate the condition. Let, for a compact set of paths, \(X\),

\[
\hat{\eta}_X(t_0, t, a, b) = |\{x \in (a, b) | \text{there exists a path in } X \text{ touching both } \mathbb{R} \times \{t_0\} \text{ and } (x, t_0 + t)\}|
\]

be a counting random variable ‘dual’ to \(\eta\) introduced before. One can then prove that \(\hat{\eta}\) and \(\eta - 1\) are equally distributed for the Brownian web \(\overline{W}\).

Sun uses this to show (see [30, p. 9]) that \((B_2)\) can be replaced by the condition

\[(E)\text{ If } X \text{ is any subsequential limit of } (X^n)_{n \in \mathbb{N}}, \text{ then } \forall t_0, t, a, b \in \mathbb{R} \text{ with } t > 0 \text{ and } a < b, \ E(\hat{\eta}_X(t_0, t, a, b)) \leq E(\hat{\eta}_{\overline{W}}(t_0, t, a, b)).\]

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Sun also uses a tightness condition \((T_1)\) stated in proposition B.1 of [12] (see [30, p. 7-8]) to prove tightness of the distributions \(\{\mu_n : n \in \mathbb{N}\}\) and so establishes the theorem by showing that \(X_n\) satisfies \((I)\), \((T_1)\), \((B_1)\) and \((E)\).

We briefly summarise some ideas of the proof: one starts by showing that the closure in \((\Pi, d)\) of the non-simple coalescing random walks \(\overline{X^n}\) is compact, thus in \(H\) (see [30, p. 18]). One uses the Arzelà-Ascoli Theorem and properties of the topology on \(\Pi\) to reduce the problem to showing that \(\overline{X^n}\) is equicontinuous when restricted to a large square \([-L, L] \times [-L, L]\), which follows because a.s. only finitely many paths of \(X^n\) are in such a square.

The verification of \((B_1)\) is relatively straightforward: key steps are that, for fixed \(\mu_n\), one must distinguish the cases whether or not \(t_0\) is in the rescaled integers and notice that if a collection of random walks has not coalesced into a single walk by time \(t\), then at least one pair of adjacent walkers has not coalesced (see [30, p. 20-22]).

The assumption \(\mathbb{E}(|Y|^5) < +\infty\) is needed to verify tightness of the set of measures \(\{\mu_n : n \in \mathbb{N}\}\) (see [30, p. 22-25]). The techniques are beyond scope here.

\((I)\) is in a sense the most interesting condition to verify. The crossing paths case is a lot more substantial then the case of simple random walks (see [30, p. 29-32]); it is clear that the condition will not simply follow from a ‘Donsker’s Invariance Principle in \(n\) dimensions’. One has to consider all three families of \(m\) independent, \(m\) ‘coalescing upon coincidence’ and \(m\) ‘coalescing upon crossing’ random walks on the rescaled integer lattice. Two collision rules, forming the second and third respectively of these families from the first, are involved, and these are noted to be continuous mappings on the paths space. So Donsker’s Invariance Principle applied to the independent paths, the continuous mapping property of weak convergence, and convergence in probability of the two different coalescing families are used to obtain convergence of \(m\) non-simple random walks starting from distinct points on the countable deterministic subset to \(m\) coalescing Brownian motions. Technical complications requiring to use interpolated and non-interpolated versions of the three families of paths at different stages of the argument prevent us from making it precise here.

The last part of the proof, verification of \((E)\), is the most difficult, since it involves the concept of stochastic ordering again and a technique of truncating paths (see [30, p. 33-37]).

We return to a fully rigorous account of further results on convergence to the Brownian web in the last two sections of this chapter.

3.4 The Poisson tree

We consider another random collection of paths for which the Brownian web is the scaling limit. The two-dimensional Poisson tree (first studied by Ferrari, Landim and Thorisson [9]) is a collection of random walks starting from the space-time points of a homogeneous Poisson process in \(\mathbb{R}^2\). The jumps and coalescence of the walks depend on the Poisson process in a way described below. We will prove that the Poisson tree converges in distribution to the Brownian web by applying the theory developed so far. This section is based on an article by Ferrari, Fontes and Wu [8].

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We start by recalling some basics (with reference to [19]): A discrete random variable $X$ has the Poisson distribution $\mathcal{P}(\mu)$ if $\mathbb{P}(X = n) = \frac{\mu^n e^{-\mu}}{n!}$, for $\mu$ a real positive constant; we include the cases $\mu = 0$ and $\mu = +\infty$ by letting $\mathcal{P}(0) = \delta_0$, $\mathcal{P}(\infty) = \delta_{\infty}$. Recall further that a two-dimensional Poisson process is a random countable set $S \subseteq \mathbb{R}^2$ such that

1. for any disjoint measurable sets $A_1, \ldots, A_n$ of $\mathbb{R}^2$, if $N(A) := |\{S \cap A\}|$ is a counting variable, then $N(A_1), \ldots, N(A_n)$ are independent;

2. $N(A)$ has the Poisson distribution $\mathcal{P}(\mu)$, where $\mu = \mu(A)$, $0 \leq \mu \leq \infty$ is a constant depending on $A$.

$S$ is called a homogeneous Poisson process if $\mu(A) = \lambda \mathcal{L}(A)$, where $\lambda \in \mathbb{R}$ is a constant (the intensity) and $\mathcal{L}$ denotes the two-dimensional Lebesgue measure. Call $\mu$ the mean measure of the Poisson process and note that if it is homogeneous then its stochastic properties are invariant under rotation and translation (spatial homogeneity).

Note that if $S$ is a homogeneous Poisson process then if it is projected onto a coordinate the projection is a (one-dimensional) Poisson process. This follows, for instance, from [19, p. 18 “Mapping Theorem"], using that $\mu(\cdot) = \lambda \mathcal{L}(\cdot)$ is $\sigma$-finite.

We proceed by defining the Poisson tree (as in [8] or [9]): Let $x = (x_1, x_2) \in \mathbb{R}^2$, $t > x_2$, $r > 0$ and

$$M(x, t, r) := \{(x'_1, x'_2) : |x'_1 - x_1| \leq r, x_2 \leq x'_2 \leq t\}$$

be a rectangle which grows with $t$. Denote the first time that $M(x, t, r)$ reaches some (another if $x \in S$) point of $S$ by

$$\tau(x, S, r) := \inf\{t > x_2 : M(x, t, r) \cap (S \setminus \{x\}) \neq \emptyset\}$$

and the unique point where $M$ hits $S$ by $\alpha(x) := M(x, \tau(x, S, r), r) \cap (S \setminus \{x\})$. Note this is a.s. well-defined because for any bounded region $B \subseteq \mathbb{R}^2$, $\mathbb{P}(N(B) = \infty) = 0$ and $\mathbb{P}(\exists s^1, s^2 \in S$ with $s^1_2 = s^2_2) = 0$. (So all the definitions below are to be understood a.s.) Finally let $\alpha^0(x) = x$ and $\alpha^n(x) = \alpha(\alpha^{n-1}(x))$ iteratively.

**Definition 26 (Poisson tree)** Let $G = (V, E)$ be the random directed graph with vertices $V = S$, edges $E = \{(s, \alpha(s)) : s \in S\}$ and direction from $s$ to $\alpha(s)$.

A graph with no cycles is called a forest, and a forest is called a tree if any two vertices are connected by a path (whether or not in the specified direction of the edges).

It is evident that there are no cycles in $G$ : Start with $s_0 \in S$ and note that it has only one outgoing edge. To ‘walk’ on $G$ in a cycle would hence involve changing direction in time at least twice. This is impossible because one cannot change from going backward in time to going forward except by going back on the same edge. So $G$ is a forest. Ferrari, Landim and Thorisson [9] proved that $G$ is also connected, so that $G$ is called the Poisson tree.

$G$ induces sets of continuous paths on $\mathbb{R}^2$ as follows: for $s = (s_0, s_1) \in S$ let $X_s$ in $\mathbb{R}^2$ be the path consisting of all edges $\{(\alpha^{n-1}(s), \alpha^n(s) : n \in \mathbb{N}\}$ of $G$, taken to be
straight lines between the vertices. With the notation of Section 2.1, and considering $G$ in $(\mathbb{R}^2, \rho)$, $X^s \in C[s_2] \times \{s_2\} \subset \Pi$.

**Definition 27 (Poisson web)** Let $X_1 := \{X^s : s \in S\}$, which we call Poisson web.

This definition evidently depends on $\lambda, r > 0$. For reasons to become clear below, choose $\lambda = \frac{\sqrt{3}}{6}$, $r = \sqrt{3}$ and rescale as usual $X$ by $\delta$ in space, $\delta^2$ in time. Then $X_\delta = \{(\delta x_1, \delta^2 x_2) \in \mathbb{R}^2 : (x_1, x_2) \in X_1\}$. Take $\delta = \frac{1}{n}$, $n \in \mathbb{N}$. It is straightforward to see that $\overline{X}_n$ is a (sequentially) compact subset of $\Pi$, where its closure is obtained by taking all boundary paths as for the simple random walks (see the remarks before Theorem 24 in Section 3.2).

So $\overline{X}_n$ is an $(H, \mathcal{F}_H)$-valued random variable (taking measurability for granted). We write $X_n$ for $\overline{X}_n$ in the following.

**Theorem 28 (Theorem 1.1. of [8])** $X_n$ converges in distribution on $H$ to the Brownian web $W$ as $n \to \infty$.

The proof of this is long and intricate because of the randomness coming from the Poisson process involved, so we focus on the main points. Because paths in the Poisson tree are non-crossing, it is sufficient to prove conditions (I), $(B''_1)$ and $(B''_2)$ (the last two stated in Section 3.2, and tightness is automatically satisfied, see the remark there) to obtain the result by the general convergence theorem (Theorem 23).

The first step is to define coalescing random walks from the Poisson tree (see [7, p. 6-7]): Fix $x = (x_1, x_2) \in \mathbb{R}^2$, let $\tau^n(x) = (\alpha^n(x))_2$ (for any $n \geq 0$) be the second coordinate of $\alpha^n(x)$. Consider the continuous-time random walk $\{\xi^\alpha(t) : t \geq x_2\}$ where $\xi^\alpha(t) = (\alpha^n(x))_1$, the first coordinate of $\alpha^n(x)$, for $t \in [\tau^n(x), \tau^{n+1}(x))$.

For each continuous-time random walk $\xi^\alpha$, if it is at $s = (s_1, s_2) \in S$ after an exponentially distributed waiting time with parameter $2\lambda r$ at chooses a point uniformly in the interval $[s_1 - r, s_1 + r]$ and jumps there.

Coalescence occurs, of course, if two random walks are at some time $t_0$ located at space points $y_1, z_1 \in \mathbb{R}$ with $|y_1 - z_1| < 2r$ and $\alpha(y_1, t_0) = s = \alpha(z_1, t_0) \in S$. We carefully rescale the random walks as follows: For any $x = (x_1, x_2) \in \mathbb{R}^2$, let $x_\delta = (\delta^{-1} x_1, \delta^{-2} x_2), \delta \in (0, 1]$. For $\xi^\alpha$, the random walk starting at $x$, the corresponding rescaled random walk is:

$$\xi^\alpha_\delta(t) := \delta \xi^\alpha(\delta^{-2} t) \text{ for } t \geq x_2,$$

that is, if $x_\delta$ is originally a point of the Poisson process, then we rescale by steps $\delta$ in space and $\delta^2$ in time.

We then replace the random walks defined above by their linearly interpolated versions to obtain a continuous path and denote it by $\xi^\alpha_\delta$.

These steps ensure that, after the rescaling, $\xi^\alpha_\delta \in X_\delta$, if and only if $x_\delta \in S$.

Define

$$\theta^\alpha_\delta := \begin{cases} 
\xi^\alpha_\delta & \text{if } x_\delta \in S \\
\xi^\alpha_\delta(\delta[\alpha(x_\delta)]_1, \delta^2[\alpha(x_\delta)]_2) & \text{otherwise}.
\end{cases}$$

We begin with two lemmas:

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Lemma 29 (Lemma 2.1. of [8]) If $\lambda = \sqrt{3}/6$, $r = \sqrt{3}$ then $\xi^\delta$ converges in distribution to $W^x$ as $\delta \to 0$, where $W^x$ is the Brownian motion started at $x = (x_1, x_2) \in \mathbb{R}^2$.

Remark: In the rest of this chapter by “weak convergence as $\delta \to 0$” we mean that weak convergence holds for every sequence $(\alpha_n)_{n \in \mathbb{N}}$ such that $\alpha_n \to 0$ as $n \to \infty$.

Proof: To apply Donsker’s Invariance Principle we can w.l.o.g. let $x$ be the origin, by homogeneity of the Poisson process. For the choice of $\lambda$, $r$, we obtain, by the above mentioned mapping property of the Poisson process, that the projection onto the time coordinate is still a one-dimensional Poisson process and that it has mean measure $\mu^*(B) = 2r\lambda \vert B \vert (B \in \mathcal{B}((\mathbb{R}))$. So for the waiting time of the random walk, we have that it has exponential distribution with parameter $2r\lambda = 1$, so mean 1. Clearly, by spatial homogeneity, the random walk increments are i.i.d and have uniform distribution on $[-r, r]$.

The random walk increment $Y$ has mean 0 and variance

$$\text{var}(Y) = \mathbb{E}(Y^2) = \int_{-r}^{r} \frac{x^2}{2r} \, dx = \frac{r^3}{6r} + \frac{r^3}{6r} = \frac{1}{3} r^2 = 1.$$ 

We use the fact that Donsker’s Invariance Principle is also valid for continuous-time random walks if the jump times have mean 1, so the lemma follows. $\square$

Regard, for $x_1, \ldots, x_m \in \mathbb{R}^2$, $m \in \mathbb{N}$, $(\xi^{x_1}, \ldots, \xi^{x_m})$ and $(\theta^{x_1}, \ldots, \theta^{x_m})$ as random walks in the product metric space $(\Pi^m, d^m)$ choosing the sup-metric as product metric.

The next lemma follows from the properties of the Poisson process:

Lemma 30 (Lemma 2.2. of [8]) Let $P_\lambda$ denote the distribution of the Poisson process, then

$$P_\lambda(\mathbb{P}^m((\xi^{x_1}, \ldots, \xi^{x_m})|\theta^{x_1}, \ldots, \theta^{x_m}) \geq \epsilon) \to 0 \text{ as } \delta \to 0$$

for $x_1, \ldots, x_m$ as above, $\epsilon > 0$ and any $m \in \mathbb{N}$.

Verification of (I):

Fix a deterministic dense countable subset $D \subset \mathbb{R}^2$ and any $y^1, \ldots, y^m \in D$. We will prove condition (I) of Theorem 23 by showing that $(\theta^{x_1}, \ldots, \theta^{x_m})$ converges in distribution as $\delta \to 0$ to coalescing Brownian motions starting at $y_1, \ldots, y_m$.

A first simplification is as follows:

Lemma 31 (Lemma 2.3. of [8]) By lemma 30 we only need to prove that $(\xi^{x_1}, \ldots, \xi^{x_m})$ converges as $\delta \to 0$ in distribution to coalescing Brownian motions starting at $y^1, \ldots, y^m$.

Remark: If $m = 1$ this is implied by Lemma 29.

This is more involved then in the case of simple coalescing random walks. More is required than simply applying Donsker’s Invariance Principle to $n$ random variables because the random walks generated from the Poisson process are not independent before coalescence. The problem is the following subtlety: If two random walks are at time $t$ within radius $2r\delta$ and both jump to the same $s = (s_1, s_2) \in S$ at time $s_2$, then their evolution between $t$ and $s_2$ is not independent although they are not coalesced.

The right strategy is hence to define an auxiliary collection of random walks in which
any two of them coalesce as soon as they are within distance $2r\delta$. We provide more
details in Section C.2 of Appendix C.

**Verification of $(B''_1)$ and $(B''_2)$:**

We proceed by verifying $(B''_1)$, $(B''_2)$ for the Poisson web $X_\delta$. Define the counting
random variable $\eta_{X_\delta}(t_0, t, a, b)$ as usual. Let $\pi_{X_\delta}(t_0, t, a, b)$ be another discrete random
variable: $\pi_{X_\delta}(t_0, t, a, b)$ is the number of distinct $y = (y_1, y_2) \in \mathbb{R} \times \{t_0 + t\}$ such that
there exists $s \in S$ with $s_2 \leq t_0$, $\xi_s(t_0) \in [a, b]$, $\xi_s(t_0 + t) = \xi_s(y_2) = y_1$, where $\xi_s$ is the
(unfinished!) continuous random walk defined earlier.

So $\eta_X$ counts interpolated, $\pi_X$ uninterpolated paths. It is then clear that we have,
for any $n$, the estimate

$$\eta_X(t_0, t, a, b) \geq n \Rightarrow \eta_X(t_0, t, a - 2r, b + 2r) \geq n \Rightarrow \pi_X(t_0, t, a - 4r, b + 4r) \geq n.$$ 

So in order to verify condition $(B''_1)$, $(B''_2)$ we only need to prove – using the space
homogeneity of the Poisson process to eliminate $\sup_{(a,b)\in R^2}$ – that

$$(B''_1) \limsup_{n \to \infty} P(\eta_{X_{\delta_n}}(0, t, 0, \epsilon) \geq 2) \to 0 \text{ as } \epsilon \to 0^+$$

$$(B''_2) \limsup_{n \to \infty} P(\pi_{X_{\delta_n}}(0, t, 0, \epsilon) \geq 3) \to 0 \text{ as } \epsilon \to 0^+$$

for any sequence $(\delta_n)_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} \delta_n = 0$.

Verifying $(B''_1)$ is straightforward: As we have convergence of a single random walk
to Brownian motion by Lemma 29, it is sufficient to consider two Brownian motions
starting at the end points of $[0, \epsilon]$ so that $(B''_1)$ is verified by showing:

**Lemma 32** \(\limsup_{n \to \infty} P(\eta_{X_{\delta_n}}(0, t, 0, \epsilon) \geq 2) = 2\phi\left(\frac{\epsilon}{\sqrt{2r}}\right) - 1, \text{ where } \phi \text{ is the standard normal distribution function.}$$

**Proof:** The calculation is exactly as in the proof of Theorem 24. \(\square\)

To verify condition $(B''_2)$ is far more complicated; we have to quote two facts,
before we can calculate an estimate.

Let $\xi = \xi^{(0,0)}$ be a random walk path as defined earlier, and let $\overline{\Pi}$ be the subset of
the path space $\Pi$ in which it takes values. Define a partial order ‘$\leq$’ on $\overline{\Pi}$ as follows.

Given $\pi_1, \pi_2 \in \overline{\Pi}$,

$$\pi_1 \leq \pi_2 \ \text{ if and only if } \pi_1(t) - \pi_1(s) \leq \pi_2(t) - \pi_2(s) \ \text{ for all } t \geq s \geq 0.$$ 

Define ‘increasing events’ in $\overline{\Pi}$: $A \subseteq \Pi$ is increasing if for $\pi_1 \in A$, $\pi_2 \geq \pi_1$ implies
$\pi_2 \in A$.

**Lemma 33 (Lemma 2.6. of [8])** $\mu_\xi$, the distribution of $\xi$ on $\overline{\Pi}$, satisfies the FKG
inequality, namely for increasing $A, B \subseteq \overline{\Pi}$, $\mu_\xi(A \cap B) \geq \mu_\xi(A)\mu_\xi(B)$.

The proof of this is by discretizing the continuous random walk and using the FKG
inequality on its increments (see [7, p. 14-15], for FKG-type inequalities see [16, ch.
2]). Let $\Delta_{2r}(t) = \xi^{(0,0)}(t) - \xi^{(2r,0)}(t)$, the difference between two random walks started
at time $t = 0$. We quote

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Lemma 34 (Lemma 2.7. of [8]) If $T = \inf\{t > 0 : \Delta_{2r} = 0\}$, then there exists a constant $c > 0$ such that $\mathbb{P}(T > t) \leq \frac{c}{\sqrt{t}}$ for any $t > 0$, $c$ depending on $r, \lambda$ only.

Finding a bound on $\limsup_{n \to \infty} \mathbb{P} (\eta_{X_{\delta n}} (0, t, 0, \epsilon) \geq 3)$, using the last two lemmas, to verify $(B''_2)$ is very elegant and uses some of the techniques discussed in this dissertation, but we cannot do better than recasting the argument given in [7, p. 18 -19], so we provide the full details in Section C.3 of Appendix C. □

Recall that in the case of the simple coalescing random walks the verification of $(B''_2)$ depended on an FKG-type correlation (cf. Section 3.2) as well. However it is a stronger FKG-property than the one used in the case of the Poisson tree, where $(B''_1)$, $(B''_2)$ are not verified at once.

3.5 Poisson tree with random ‘radius of attraction’

In this last section on the convergence to the Brownian web we consider an example for which, to the best of our knowledge, convergence has not been verified in the literature. B. Micaux, in his PhD thesis [22], sets the scene by presenting an extension of the Poisson tree model: call the quantity $r$ the radius of attraction. Instead of fixing $r$ for all points of the Poisson process $S$, we wish to vary $r$ (cf. [22, p. 5-6, 61-66]). Certain restrictions on how to do so have to be imposed to well-define an object similar to the Poisson tree but with random $r$.

In Micaux’s thesis the Poisson tree is considered in the context of stochastic flows driven by the noise of the Poisson process. Here we will discuss the problem of whether the ‘Poisson tree with random radius’ still converges to the Brownian web.

![Figure 3: Poisson tree with random radius r. Source: [22, p. 5], slightly modified.](image)

Fix $\lambda \in \mathbb{R}$. Micaux considers the Poisson tree with random radius as a Poisson process $P$ on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$ with mean measure $\mu = \lambda (\mathcal{L}(\cdot) \otimes \mathcal{L}(\cdot)) \otimes \sigma$. $\mathcal{L}(\cdot)$ denotes the Lebesgue measure.
measure on $\mathbb{R}$ and $\mathbb{R}_+^*$, the third component, is the range of the radius of attraction $r$, added to the Poisson process generating the tree of Section 3.4. Micaux can then show ([22, p. 63-66]) that continuous-time random walks generated from this Poisson process $P$ can be well-defined if $\sigma$ obeys a finiteness restriction: $\sigma$ has finite mean in the sense that $\int_0^\infty x d\sigma(x) < \infty$.

For our purpose, discussing the convergence to the Brownian web, we change Micaux’s definition slightly, instead considering the Poisson tree with variable radius of attraction as a Poisson process on $\mathbb{R} \times \mathbb{R} \times [0, R]$ with a mean measure that has Lebesgue measure even in the third component, i.e. $\mu = \lambda \mathcal{L}(\cdot) \otimes \mathcal{L}(\cdot) \otimes \mathcal{L}(\cdot)$. We will prove the analogue of Lemma 29.

Define the linearly interpolated random walk $\xi^x_\delta$ generated from the Poisson process as in the previous section, except that it now additionally depends on the random quantity $r \in [0, R]$.

**Lemma 35**: If $R = \sqrt{6}$, $\lambda = \frac{1}{6}$, then any linearly interpolated random walk $\xi^x_\delta$ converges in distribution to $B^x$ as $\delta \to 0$, where $B^x$ is a Brownian motion started at $x$.

**Proof**: To apply Donsker’s Invariance Principle we must ensure that the waiting times of $\xi^x_\delta$ are exponentially distributed with mean 1 and that the increments of $\xi^x_\delta$ are i.i.d. and have mean 0 as well as variance 1.

Note that because of the space homogeneity of the Poisson process we can assume w.l.o.g. $x = (0, 0)$. Note also that the first point of $P$ to which $\xi^{(0,0)}$ will jump is in the region $(0, \infty) \times \Delta$ where $\Delta = \{((x, y) \in \mathbb{R} : |x| \leq y) \cap ([-R, R] \times [0, R])\} \subseteq \mathbb{R} \times \mathbb{R}_+^*$. Note finally that by the ‘Mapping theorem for Poisson processes’ (see [19, p. 18-20]), we know that projecting the Poisson process restricted to $(0, \infty) \times \Delta$ onto the time axis still gives a one-dimensional Poisson process. So the waiting time has exponential law. If we denote by $\pi$ the projection, then we have the relationship

$$\mu^*(B) = \mu(\pi^{-1}(B)) = \lambda \mathcal{L}(B) \cdot \mathcal{L}(\Delta).$$

where $\mu^*$ is the mean measure of the one-dimensional Poisson process and $\mu$ as above. However $\mathcal{L}(\Delta) = R^2$, therefore the waiting times are distributed exponentially with parameter $\lambda R^2$. So the mean jump time is, by property of the exponential distribution,

$$\frac{1}{\lambda R^2} = \frac{1}{\frac{1}{6}(\sqrt{6})^2} = 1.$$

By spatial homogeneity of the Poisson process $P$ we only need to prove the properties of the distribution of the increment $Y$ of $\xi^{(0,0)}$ for the first jump – all increments will be i.i.d.. Let $p \in [-R, R]$ be the space point of the Poisson process $P$ to which $\xi^{(0,0)}$ will jump, $r_p$ its radius. By the homogeneity again, $(p, r_p)$ has uniform distribution on $\Delta$. Hence we find the probability density function of the distribution of $Y$ by calculating

$$\frac{1}{R^2} \int_{y=|x|}^R dy = \frac{1}{R^2} (R - |x|).$$

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So the random walk increment has expectation:
\[
E(Y) = \frac{1}{R^2} \int_{-R}^{R} x(R - |x|) \, dx = \frac{1}{R^2} \left( (\frac{1}{2}x^2R - \frac{1}{3}x^3)^R_0 + (\frac{1}{2}x^2R + \frac{1}{3}x^3)_0^{-R} \right) = 0.
\]

For the variance we obtain:
\[
\text{var}(Y) = \text{E}(Y^2) = \frac{1}{R^2} \int_{-R}^{R} x^2(R - |x|) \, dx = \frac{1}{R^2} \left( (\frac{1}{3}x^3R - \frac{1}{4}x^4)^R_0 + (\frac{1}{3}x^3R + \frac{1}{4}x^4)_0^{-R} \right)
= \frac{2}{3}R^2 - \frac{1}{2}R^2 = 1. \quad \Box
\]

**Remark:** It is clear that this follows because we can use the parameters \( \lambda \) and \( R \) to obtain the required jump time and variance of the increment. One could hence also follow Micaux’s approach and achieve this by considering the positive real line for the third component of the state space of the Poisson process and a general measure in the third component with some explicit finiteness restriction on it. Bounding the random radius on a finite interval however is necessary for proving the convergence to the Brownian web.

We would now like to prove the following:

**Conjecture 36** For \( \lambda = \frac{1}{6} \), \( R = \sqrt{6} \), the Poisson web with random radius, \( X_5^\delta \), obtained from the random walks generated by the Poisson process \( P \) analogously to the Poisson web defined in Section 3.4, converges in distribution to the Brownian web as \( \delta \to 0 \).

**Discussion:** This is a matter of checking that the proof of Theorem 28 still follows through. We just proved the analogue of Lemma 29. Obviously the rescaling of the random walks works similarly and in Lemma 30 nothing changes.

Concerning Lemma 44, we note that Micaux ([22, ch. 3.5.]) proved that stochastic flows originating from this model are still coalescing, so it should be true also for the random walks. We note further that the arguments in Lemma 45 itself do not need to be changed for this modified problem. However the definition of \( \delta \)-coalescence preceding the lemma needs modification as it depends on \( r \). Having chosen to have the random radius in \([0, R]\), we hope to fix this by modifying the \( \delta \)-coalescence by choosing the critical distance to be \( 2R\delta = 2\sqrt{6}\delta \).

Further, we checked that the relation between \( \eta \) and \( \overline{\eta} \) preceding Lemma 45 holds easily if \( r \) is replaced by \( R \). Lemma 32 immediately holds again, for it does not depend on the modification and we believe the same is true for Lemma 33 which also does not depend on \( r \) but only on the distribution of a single random walk.

Something similar can be expected for Lemma 34, which depends on the distribution of two random walks only. However the details of its proof in [7] suggest a modification will be complicated.

Finally the derivation of an upper bound to verify \( (B_n^\eta) \) depends in at least one place on a fixed \( r \), namely when bounding \( n \) above, and we have not yet managed to see how to modify this.

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3.6 Conclusion

A related open problem of some significance in population genetics [A. Etheridge, pers. comm.] consists in understanding a different way of extending on the Poisson tree model (now with fixed $r$ again). Suppose that the random walks of the Poisson tree are generated from the Poisson process as in Section 3.4 except that every time a random walker gets within radius $r$ of a point of the Poisson process it has (fixed) probability $p$ of jumping to that point and probability $1 - p$ of not jumping there. One would like to know what the scaling limit of this extension is.

We remark that it is not clear whether the configuration of random walks is still a tree, as opposed to merely a forest. However we know that the random walk increment is still distributed uniformly on $[x - r, x + r]$ if the walker starts on $x$. The mean jump time is now slower than exponential with mean $\frac{1}{2\lambda}$, but the walker will jump after finite time a.s.. So one could rescale this and apply Donsker’s Invariance Principle again.

A further difficulty with this model is that some random walk paths will cross, because if two walkers are within distance $r$ one worker may and the other may not jump to a certain point of the Poisson process. Therefore one would need to employ arguments as in [30] (see Section 3.3) to attempt a proof of convergence to the Brownian web.

3.6 Conclusion

We conclude this dissertation with a summary of what we have achieved in this work and with a few remarks on ramifications of the project.

We introduced the Brownian web as the scaling limit on $\mathbb{R} \times \mathbb{R}$ of the simple coalescing random walks in discrete space and time. In Chapter 1 we set the scene for a discussion of the Brownian web by assembling the necessary background. We discussed Donsker’s Invariance Principle, the construction of finitely many coalescing Brownian motions, and we gave a summary of Arratia’s original viewpoint on the object, to highlight the technical complications of defining the Brownian web. We outlined, in Appendix A, the connection between the voter model and coalescing random walks as the broad motivation for studying the Brownian web.

In the second chapter we introduced the setting chosen by Fontes et al. [12] for the Brownian web as a random variable into a metric space of compact sets of paths. After examining that topology we constructed in detail the Brownian web and provided two characterisations for it (Theorems 9, 17): in addition to the requirement that the paths starting from a countable dense subset must be distributed as coalescing Brownian motions, one can either define the Brownian web to be the closure of those paths or provide a different minimality condition. We discussed (in Appendix B) the Double Brownian web, the scaling limit of forward and backward coalescing random walks, and stated a result about how many in- and outgoing paths there can be at points in the Brownian web (Theorem 42).

Chapter 3 provided a general convergence theorem (Theorem 23) for the Brownian web, as well as particular results: we proved that the simple coalescing random walks converge to the Brownian web in this setting – the analogue of Donsker’s Invariance
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Principle – and gave a brief summary of Sun’s work on the case of crossing random walk paths.

We looked in greater detail at the case of convergence of the Poisson tree to the Brownian web and considered a recent extension to this: that of a Poisson tree with random radius of attraction. We added a calculation of our own to the existing literature and checked some more steps that suggest that one should be able to prove that even that extension converges to the Brownian web. We finished our study of the Brownian web by giving, in Appendix D, a sketch summary of the very latest results on the topic, notably Sun’s and Swart’s ‘Brownian net’ and the alternative viewpoint of Norris and Turner.

There are three directions in which this work could have been extended. First, we were unfortunately unable to include the very important contribution of Tóth and Werner [32] who used, before the work of Fontes et al. [12], the Brownian web as an auxiliary object to construct a continuous ‘self-repelling motion’ belonging to the theory of self-interacting processes. It was impossible to treat both that topic and that of the Poisson tree and we found that the latter fitted better with the other material in this dissertation.

Second, one should be interested in (putative) applications of the recent theory of the Brownian web. Several authors comment on possible applications to modelling drainage networks (to be found for instance in [17] and the references therein), we suggest this could be interesting material to be dealt with in another dissertation, as would be a more detailed examination of the latest developments touched upon in Appendix D.

Finally, we much regret that no further rigorous treatment of the open problems discussed in Section 3.5 could be carried out, however we may hope to undertake the further work to be done in this respect after completion of this project in a different context.

From a personal perspective, I thought it exciting that the topic was of so much recent interest, with some important publications ([22], [31]) only appearing after I started working on it. Devoting this dissertation to the Brownian web has proved a challenging, stimulating and very rewarding task.

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Appendix A

Voter models and their connection with coalescing random walks

This appendix is a digression from the main strand of the dissertation as we do not treat here material directly concerned with the Brownian web. Instead we sketch the connection of the topic of this dissertation with the theory of interacting particle systems.

The voter model is, along with the celebrated stochastic Ising model, originating from statistical mechanics, the most important object of study in that field. We give a succinct summary (following [6], [21] and [30]).

We describe the two state $d$-dimensional discrete time case first: The state space of this voter model is $\{0, 1\}^\mathbb{Z}^d$. Each site $x \in \mathbb{Z}^d$ stands for an individual, a ‘voter who holds exactly one of two political opinions’, 0 or 1; denote its ‘opinion’ at time $n$ by $\eta_n(x)$.

This becomes a stochastic, interacting particle system by supposing that at any time $n$ every voter forgets his opinion and randomly adopts the opinion of one of his $2d$ neighbours with equal probability. Express this by letting $X$ denote the increment of a simple $\mathbb{Z}^d$-valued random walk; let $(X_{n,x})_{x \in \mathbb{Z}^d, n \in \mathbb{N}}$ be i.i.d. random variables distributed as $X$ and update the particle system according to the rule

$$\eta_n(x) = \eta_{n-1}(x + X_{x,n}).$$

A slightly different case of a voter model is where only one voter - instead of every voter - changes his opinion at a particular time $n$, and we consider a finite square in $\mathbb{Z}^d$ identifying it to a discrete torus. Interestingly in that case it is fairly straightforward to prove, using the martingale convergence theorem, that given any initial configuration of ‘opinions’ this latter voter model converges to and reaches in finite time a state in which either all individuals ‘vote’ 1 or all ‘vote’ 0 (see [20, p. 216-217]).

In the infinite voter model described above (and in a continuous time version of it) the limit behaviour is more complicated. Obviously the states $\eta \equiv 1$ for all $x$ and $\eta \equiv 0$ for all $x$ still represent stationary distributions, and in $d \leq 2$ the system converges to complete consensus starting from any initial configuration. In $d > 2$, there are, however, also non-trivial stationary distributions (see [6, p. 2], [21, ch. 5]), an analysis of which is an essential part of the theory of interacting particle systems.

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What has the voter model to do with the Brownian web? There is a duality relation between the voter model and the coalescing random walks, which has been exploited to obtain results about the voter model. So we finish this chapter by outlining the relationship between the voter model and coalescing random walks.

The duality is described in much generality in [21, p. 157 ff.]; we follow (in part) the simpler approach of [6, p. 21-24]. Consider for simplicity the continuous-time voter model in \( \mathbb{Z}^d \). Let \( \{ T^x_n, n \geq 1, x \in \mathbb{Z} \} \) be independent one-dimensional Poisson processes with rate 1 and \( (X^x_{n,x})_{x \in \mathbb{Z}^d, n \in \mathbb{N}} \) i.i.d. as above. Then at time \( T^x_n \) the voter at \( x \) decides to change her mind and adopt the opinion of her neighbour at site \( x + X^x_{n,x} \). The graphical representation of certain classes of particle systems (see [21, p. 172 ff.]), well explained in this simple case in [6], helps to see the relationship to coalescing simple random walks.

For each \( x \) and \( n \), draw an arrow from \((x + X^x_{n,x}, T^x_n)\) to \((x, T^x_n)\) and mark \((x, T^x_n)\) by \( \delta \).

![Graphical representation of the one-dimensional voter model.](image)

If one considers the opinion at a particular site at time 0 then it may ‘branch’ in the sense that another site adopts it and the original site keeps it, or it may be annihilated if the original site adopts the opinion of a neighbour first. In that sense tracing opinions backwards in time leads to coalescing simple random walks.

To make this precise (see [6, p. 21-22]), we say there is a path from \((x, 0)\) to \((y, t)\) in \( \mathbb{Z} \times \mathbb{R} \) if there is a sequence of times \( s_0 = 0 < s_1 < s_2 < \ldots < s_n < s_{n+1} \) and spatial locations \( x = x_0, x_1, \ldots, x_n = y \) such that:

1. For \( i = 1, 2, \ldots, n \) there is an arrow from \( x_{i-1} \) to \( x_i \) at time \( s_i \),
2. the vertical segments \( \{x_j\} \times (s_i, s_{i+1}), i = 0, 1, \ldots, n \) do not contain any \( \delta \)s.

So if there is a path between \( x \) and \( y \), then \( y \) at \( t \) has the same opinion as \( x \) at 0. For \( A, B \subseteq \mathbb{Z} \), define

\[
\xi^A_t = \{ y \mid \text{for some } x \in A \text{ there is a path from } (x, 0) \text{ to } (y, t) \}\]

and

\[
\hat{\xi}^B_t = \{ x \mid \text{for some } y \in B \text{ there is a path from } (y, \hat{0}) \text{ to } (x, \hat{t}) \}\]

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where we reverse the direction of the arrows and reverse time by taking \( \hat{s} = t - s \). However, the \( \hat{\xi} \) just defined depend “too much” on the random realisations of the voter model so that one gets a too irregular behaviour of the dual process: for instance, in the random realisation shown in figure 4, \( \hat{\xi}^{\{0\}}_t = \{1\} \) for \( t_1 < t < t_5 \) and \( \hat{\xi}^{\{0\}}_t = \{-2\} \) for \( t > t_5 \).

To get around this let

\[
\tilde{\xi}^B_t = \{ x \mid \text{for some } y \in B \text{ there is a path from } (y, \hat{0}) \text{ to } (x, \hat{t}) \}
\]

where we fix \( t \), letting time evolve as \( \hat{s} = t - s \) for \( t \) fixed. So for \( \tilde{\xi}^B_t \) imagine that time did not start at \( t = 0 \) but goes back to \(-\infty\), to have a regular behaviour of the dual process.

It is immediate that we have the relationship

\[
\{ \xi^A_t \cap B \neq \emptyset \} = \{ \tilde{\xi}^B_t \cap A \neq \emptyset \}.
\]

Then as \( \hat{\xi}^B_t, \tilde{\xi}^B_t \) have the same distribution we also have

\[
P(\{ \xi^A_t \cap B \neq \emptyset \}) = P(\{ \tilde{\xi}^B_t \cap A \neq \emptyset \}).
\]

One is sometimes only interested in the genealogies of opinions of type 1. In particular, if \( A \) is the set of sites with opinion 1 at time \( t = 0 \) and \( B \subseteq \mathbb{Z} \times \{t\} \), for some \( t > 0 \), then rewriting the last equation for that case gives

\[
P(\exists x \in B \text{ such that } x \text{ has opinion 1}) = P(\exists y \in \tilde{\xi}^B_t \text{ such that } y \text{ has opinion 1}).
\]

It is finally evident that, if \( B \) is the set of sites with opinion 1 at time \( t \) then the \( \tilde{\xi}^B_t \) are distributed as continuous-time random walks backwards in time. So we have deduced the duality relation between the voter model and coalescing random walks. It holds likewise for any \( \mathbb{Z}^d \) for \( d > 1 \). That duality relation has for instance been exploited in an influential paper by T. Cox [5]: it is used to examine the asymptotics of the consensus times of the voter model on the discrete torus (see above) in continuous time as the extent of the finite square in \( \mathbb{Z}^2 \) (the torus is formed of) tends to infinity.

This discussion shows that to a certain extent the Brownian web is also a useful object of study in connection with the voter model. In particular when considering extensions and modifications of the voter model the Brownian web has recently been used to understand these (for instance in [10]).
Appendix B

The Double Brownian web and “types of point” in the Brownian web

In this appendix we discuss further theory of the Brownian web which is essential to understanding the object and builds on the results of Chapter 2: the Brownian web can be extended to a ‘Double Brownian web’ of paths running forwards and backwards, and a classification of the ‘types of points’ occurring in it provides further insight into its structure. Few statements made in this Appendix, which is based on [28] and [12, section 5], will be proved because the proofs are beyond the scope of this work.

The Double Brownian web combines the Brownian web with a dual Brownian web of coalescing Brownian motions ‘starting everywhere, but moving backwards in time’: Let us reconsider the coalescing random walks starting on the even points of the integer lattice presented in the introduction.

Figure 5: Forward and backward simple coalescing random walks. Source: [12, p. 19]
Construct ‘dual’ (not quite in the proper graph-theoretic sense) random walks from the odd points in the integer lattice ‘reflecting’ off the forward random walks, so that forward and backward random walks never cross, as shown in Figure 5. In fact it is rather simple to construct the dual system: If \((x, t), (x + 1, t + 1)\) is an edge in the random realisation of the forward walks, then \((x, t + 1), (x - 1, t)\) is an edge in the backwards system of walks; and if \((x, t), (x - 1, t + 1)\) is an edge in the forwards system of walks, then \((x, t + 1), (x + 1, t)\) is in the backward system ([28, p. 544]).

This works because ‘directly above a forward line, there can never be another forward line’. The forward and backward system are of course ‘symmetric’ in their distribution, so, heuristically, the forward system of random walks has the Brownian web as a scaling limit and the backward system a scaling limit that is a Brownian web run backwards in time, which we label ‘dual Brownian web’. We call the two objects combined the ‘double Brownian web’ and expect that in this limiting object forwards and backwards running paths will never cross each other.

The construction of the double Brownian web is much analogous to the construction of the Brownian web given in section 3.2. but not possible without a result of Soucaliuc, Tóth and Werner in [28].

Start with countably many i.i.d. Brownian motions \(B_1, B_1^b, B_2, B_2^b, \ldots\) and construct forwards and backwards independent Brownian motions \(W_j, W_j^b\) starting from \((x_j, t_j) \in D\), \(D\) a countable dense subset of \(\mathbb{R}^2\), as follows:

\[
W_j(t) = x_j + B_j(t - t_j), \quad t \geq t_j
\]

\[
W_j^b(t) = x_j + B_j^b(t_j - t), \quad t \leq t_j
\]

The difficulty is to turn this into a set of forward and backward coalescing Brownian motions in which forward and backward paths do not cross each other, which then turns out to be the limit of the forwards and backwards random walks.

Soucaliuc’s, Tóth’s and Werner’s idea is to show that “a Brownian motion reflected on an independent time-reversed Brownian motion is again a Brownian motion” (see [29]) and that this can be extended to families of forward and backward running coalescing Brownian motions (here ‘reflected’ has a special sense, see below.)

Let us make this precise in the simplest possible case (see [29, p. 117-18]). If \(f, g \in C[0, T]\) and \(g(0) > f(0)\) define the forward reflection of \(f\) on \(g\) as

\[
\tilde{f}_g(t) = f(t) - \sup_{0 \leq s \leq t} (f(s) - g(s))_+,
\]

if \(f(0) > g(0)\) define it as

\[
\tilde{f}_g(t) = f(t) + \sup_{0 \leq s \leq t} (f(s) - g(s))_-.
\]

This definition captures that \(\tilde{f}_g\) has the same path as \(f\) as long as it stays away from \(g\), but gets a push (in the direction it came from) that prevents it from crossing \(g\) whenever it hits \(g\). This is meant by ‘reflection’ in this context. So in the case \(g(0) > f(0), \tilde{f}_g \leq g\) and \(f_g - f\) is constant on every interval where \(f_g \neq g\).
Similarly define the backward reflection with time running backwards: If \( f(T) < g(T) \) then

\[
g^f = g(t) + \sup_{t \leq s \leq T} ((g(s) - f(s))_+),
\]

similarly in the case \( f(T) > g(T) \). See [29, p. 512] for a picture.

The result of Soucaliuc, Tóth and Werner can be stated as follows:

**Proposition 37 (after Proposition 1 of [29])** Suppose that \((U_t, V_t)_{t \in [0, T]}\) is a pair of independent processes such that \((U_t)_{t \in [0, T]}\) is a Brownian motion. starting from \(x_0\), \((V_{T-t})_{t \in [0, T]}\) is a Brownian motion started from \(y_T\) \((V_t\) is a time-reversed Brownian motion). Then the laws of \((U, V^U)\) and \((U, V)\) are identical, and in particular \(U\) has the same law as \(U^V\), so \(U^V\) is a Brownian motion.

Note that this is a.s. well-defined, because a.s. \(x_0 = U_0 \neq V_0\), and \(U_T \neq V_T = y_T\), the cases for which we have not defined reflection above do not occur a.s..

We have made precise the central idea that goes into the construction of the Double Brownian web, but in fact, Soucaliuc, Tóth and Werner prove something more general. We need to introduce some more notation: let \(B^p\) be a forward Brownian motion if it is defined on \([T(j), \infty)\) and a backward Brownian motion if it is defined on \((-\infty, T(j)]\). Soucaliuc et al. then define coalescence \(C\) of a forward function \(f\) with \(p\) forward Brownian motions, as we did earlier, and they achieve to define (details omitted, to be found in [28, p. 524-27]) an operation \(R\) of ‘reflecting’ a function \(f\) on \(p\) backward Brownian motions.

Under some technical conditions, one can combine coalescence and reflection to define an operation on the space of continuous paths with variable endpoints by

\[
CR(f; g_1, \ldots, g_p) = C(R(f, g_{j(1)}, \ldots, g_{j(k)}); g_{i(1)}, \ldots, g_{i(l)}),
\]

where the indices \(j\) and \(i\) sort the \(p\) functions \(g_1, \ldots, g_p\) into backward and forward paths. This operator is then shown to represent that ‘\(f\) is first reflected at the backward functions and then coalesced with the forward functions’. It can be defined analogously for \(f\) running backwards.

Due to technical complications with the idea we only quote the main result of [28]:

**Theorem 38 (after [28], Theorem 8)** If \(B^1, \ldots, B^p\) is a family of \(p\) Brownian motions in \(C((-\infty, \infty))\) with specified starting point and direction \((t_i, a_i, \epsilon)(i)\), define

\[
(C^1, \ldots, C^p) = CR(B^1, \ldots, B^p)
\]

inductively by

\[
B^1 = C^1 \quad C^i = CR(B^i; B^1, \ldots, B^{i-1}), \quad \text{for} \ i \in \{2, \ldots, p\}
\]

where we assume \((t_i, a_i, \epsilon_i) \neq (t_j, a_j, \epsilon_j)\) for all \(i \neq j\) and \(\epsilon(j) = \pm 1\) denotes the direction of the Brownian motion.

Let \(\sigma \in S_p\). Then: \(B = (B^1, \ldots, B^p)\) and \(B^\sigma = (B^{\sigma(1)}, \ldots, B^{\sigma(p)})\) are such that the above definition of \(CR\) is a.s. well-defined. Whenever it is well-defined

\[
(CR(B^1, \ldots, B^p))^\sigma = CR(B^{\sigma(1)}, \ldots, B^{\sigma(p)})
\]

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in law and, if \((X^1, \ldots, X^n) = CR(B^1, \ldots, B^p)\) then, in law, \(X^j = B^j\) for all \(j \in \{1, \ldots, p\}\), so \(X^j\) is a Brownian motion.

The proof is beyond the scope of this dissertation, we merely remark that for \(p = 2\), restricting to a closed bounded interval, this gives the last proposition as a special case.

We now come back to the double Brownian web. For the previously introduced forward and backward Brownian motions \(W_1, W_2, W_b^1, W_b^2, \ldots\) Soucaliuc’s, Tóth’s and Werner’s result allows us to turn these into coalescing reflecting Brownian motions, setting \(\tilde{W}_1 = W_1, \tilde{W}_b^1 = W_b^1\) (no interaction as \(W_1, W_b^1\) run into different directions from the same starting point), and then

\[
\tilde{W}_n = CR(W_n; \tilde{W}_1, \tilde{W}_b^1, \ldots, \tilde{W}_{n-1}, \tilde{W}_{n-1})
\]

\[
\tilde{W}_b^n = CR(W_b^n; \tilde{W}_1, \tilde{W}_b^1, \ldots, \tilde{W}_{n-1}, \tilde{W}_{n-1})
\]

Call \(W_n^D := \{\tilde{W}_1, \tilde{W}_b^1, \ldots, \tilde{W}_n, \tilde{W}_{b^n}\}\) the coalescing reflecting forward and backward Brownian motions.

By the above theorem \(\{\tilde{W}_1, \ldots, \tilde{W}_n\}, \{\tilde{W}_b^1, \ldots, \tilde{W}_b^n\}\) are respectively forward and backward Brownian motions, so that \(\{\tilde{W}_1, \tilde{W}_2, \ldots\}\) and \(\{\tilde{W}_b^1, \tilde{W}_b^2, \ldots\}\) are forward and backward Brownian web skeletons.

Define, for the backward paths, spaces \((\Pi^b, d^b), (H^b, d_{H^b})\) similar to the metric spaces defined at the beginning of the chapter.

Form the space \(H^2 = H \times H^b\) with product metric

\[d_{H^2} = \max\{d_H(K_1, K_2), d_{H^b}(K_1, K_2^b)\}\]

Then define

\[
W_k^D = \{\tilde{W}_1, \ldots, \tilde{W}_k\} \times \{\tilde{W}_b^1, \ldots, \tilde{W}_b^k\}
\]

and \(W^D = \bigcup_k W_k^D = \{\tilde{W}_1, \ldots, \tilde{W}_k, \ldots\} \times \{\tilde{W}_b^1, \ldots, \tilde{W}_b^k, \ldots\}\), the Double Brownian web skeleton based on \(D\). Finally define:

**Definition 39 (the Double Brownian web)** The Double Brownian web is the set

\[
\overline{W}^D = \overline{\{W_1, W_2, \ldots\}} \times \overline{\{W_b^1, W_b^2, \ldots\}}
\]

where the closures are taken in \(\Pi, \Pi^b\) respectively.

Note that the product space is chosen for the Double Brownian web in order to have \(\overline{W} = \overline{\{W_1, W_2, \ldots\}}, \overline{W_b} = \overline{\{W_b^1, W_b^2, \ldots\}}\) a forward and a backward Brownian web in the ‘natural’ spaces \(H\) and \(H^b\) respectively.

The Double Brownian web is then characterised analogously to the Brownian web (see [12, p. 22-25] for details, which are not hard). We state without proof the characterisation of the double Brownian web, which concludes our discussion of it:

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Theorem 40 (after theorem 5.8. of [12]) Almost surely $\bar{W}$, $\bar{W}^b$ are $(H, \mathcal{F}_H)$-, $(H^b, \mathcal{F}_{H^b})$-random variables respectively, so a.s. the double Brownian web, $\bar{W}^D$ is an element of $H^2$ and it is $\mathcal{F}_{H^2} = (\mathcal{F}_H \times \mathcal{F}_{H^b})$-valued random variable. The Double Brownian web exists and its distribution is uniquely determined by the following three properties:

1. From any deterministic point $(x, t) \in \mathbb{R}^2$ there is a.s. a unique forward and a unique backward path starting.

2. For any deterministic $(y_1, s_1), ..., (y_n, s_n)$, the forward and backward paths in $\bar{W}^D$ starting from that set are distributed as coalescing reflecting forward and backward Brownian motions.

3. The distribution of $\bar{W}^D$ does not depend on the countable dense set it was constructed from, and for any other $D'$ countable and dense in $\mathbb{R}^2$, $\bar{W}^D$ is equal in distribution to the closure of a Double Brownian web skeleton constructed from that set.

We finally turn briefly to another result characterising the Brownian web (that much extends the content of Theorem 7 we gave in Section 1.3): call two space-time points of the same ‘type’ if they have the same number of paths going out and coming in. We would like to know what types of points occur in the Brownian web and ‘how many’ of them there are (see [12, p. 25-33] for full details, here we follow [31, p. 21-22]).

Definition 41 (type of point) An incoming path at $(x_0, t_0) \in \mathbb{R}^2$ is a path $\pi$ that starts at $t < t_0$ and $\pi(t_0) = x_0$. Incoming paths $\pi$, $\pi'$ are equivalent if $\pi(s) = \pi'(s)$ on $(u, \infty)$ for some $u < t_0$. Let $m_{in}(x_0, t_0)$ denote the number of equivalence classes of incoming paths and analogously define $m_{out}(x_0, t_0)$ as the number of paths through $(x_0, t_0)$ that are disjoint for some time immediately after $t_0$. Then the pair of natural numbers $(m_{in}, m_{out})$ is the type of a point in the Brownian web.

For the dual Brownian web we similarly define $m_{in}^b(x_0, t_0)$ and $m_{out}^b(x_0, t_0)$. Moreover let $S_{x,y}$ denote the set of points in $\mathbb{R}^2$ of type $(x, y)$.

One can prove that in the double Brownian web a.s., for every $(x_0, t_0) \in \mathbb{R}^2$,

\[ m_{out}(x_0, t_0) = m_{in}^b(x_0, t_0) + 1 \] and \[ m_{in}(x_0, t_0) = m_{out}^b(x_0, t_0) - 1 \]

The main result is the following theorem, which Fontes et al. attribute partly to Tóth and Werner ([32]).

Theorem 42 (after Theorems 5.13., 5.15 of [12] and Lemma 3.3 of [31]) For the Brownian web, almost surely, every point in $\mathbb{R}^2$ is of one of the following types: $(0, 1)$, $(0, 2)$, $(0, 3)$, $(1, 1)$, $(1, 2)$, $(2, 1)$. Almost surely, $S_{0,1}$ has full Lebesgue measure in $\mathbb{R}^2$, $S_{1,1}$, $S_{0,2}$, $S_{1,2}$ are uncountable and $S_{2,1}$, $S_{0,3}$ are countable and dense in $\mathbb{R}^2$.

Remark: One can further determine the Hausdorff dimension of the sets $S_{1,1}$, $S_{0,2}$, $S_{1,2}$. One can also determine the cardinality of the sets when restricted to any strip of space at a particular time, $\mathbb{R} \times \{t\}$, and also extend this classification to the double Brownian web. The arguments are interesting and reveal much about the Brownian web (for instance one uses repeatedly the construction of the object from the dense countable subset of $\mathbb{R}^2$ and Proposition 21. We refer the reader to [12, 31, 32].
Appendix C

Proofs of some results of Chapter 3

C.1 Proof of Theorem 23

We maintain the notation of Section 3.1 and give a summary of the proof of Theorem 23.

The theorem is proved by showing that for any limit \( \mu \) of a weakly converging subsubsequence of \( \{ \mu_n : n \in \mathbb{N} \} \), we have that \( \mu = \mu_W \) is the distribution of the Brownian web.

We first explain why this strategy works: Tightness of \( \{ \mu_n : n \in \mathbb{N} \} \) implies that every subsequence of \( \{ \mu_n : n \in \mathbb{N} \} \) has a subsequence converging to some \( \mu \). This is because by Prohorov’s theorem (see [20, p. 248-49] or [4, p. 59-60]), if \( G \) is any family of probability measures which is tight, it is weakly relatively sequentially compact in a suitable metric space of measures. So for \( G = \{ \mu_n, n \in \mathbb{N} \} \), by definition of sequential compactness, a subsequence of every sequence \( (\mu_m)_{m \in \mathbb{N}} \subset G \) converges to some limit in the closure of \( G \).

It is hence sufficient to prove the above to establish Theorem 23 because of the following elementary argument: if \( (\alpha_m) \) is any sequence in a metric space of which every subsequence \( (\alpha_{m_l}) \) has a convergent subsubsequence \( \alpha_{m_{k_l}} \to \alpha \) as \( l \to \infty \), then \( \alpha_m \to \alpha \) as \( m \to \infty \). (This is since, if assuming the contrary, there exists \( \epsilon > 0 \) such that there exists \( \alpha_n \in D(\alpha_m, \alpha) > \epsilon \); by assumption \( \alpha_m \) has a subsequence \( \alpha_{m_{k_l}} \) converging to \( \alpha \). Contradiction.)

We continue by elaborating on the key argument of the proof: Let \( \mu \) be any subsequential limit of \( (\mu_n)_{n \in \mathbb{N}} \) and let \( X \) (possibly on a different probability space) be the \( (H, \mathcal{F}_H) \)-valued random variable with distribution \( \mu \). We will, from somewhat stronger assumptions then the convergence conditions, infer that \( \mu = \mu_W \). That accomplished, the last step in the proof is then to reconcile the original conditions with the ones stated in the next lemma:

**Lemma 43 (Lemma 6.1. of [12])** Assume condition (1) of theorem 17: For a fixed dense countable subset \( D \) of \( \mathbb{R}^2 \), each \( y \in D \), let \( \theta^y \in X \) be some single random path starting at \( y \), then the \( \theta^y \) are distributed as coalescing Brownian motions. Assume further

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\( (B'_1) \forall \beta > 0 \sup_{t \geq \beta} \sup_{x \in \mathbb{R}} \mu(|N_{t_0,t}([a - \epsilon, a + \epsilon])| > 1) \to 0 \) as \( \epsilon \to 0^+ \)

\( (B'_2) \forall \beta > 0 \frac{1}{\epsilon} \sup_{t \geq \beta} \sup_{x \in \mathbb{R}} \mu(N_{t_0,t}(\{a - \epsilon, a + \epsilon\}) \neq (N_{t_0,t}^+(\{a - \epsilon, a + \epsilon\}) \cup N_{t_0,t}^-(\{a - \epsilon, a + \epsilon\})) \to 0 \) as \( \epsilon \to 0^+ \)

Then \( \mu \) is the distribution of the Brownian web.

**Proof of Lemma:** Note that \( X \) is a compact set of paths, so by (1) and \( (B'_1) \) (and using Proposition 21) it also satisfies conditions (1) and (2) of Theorem 9, that is for any deterministic point there is a unique paths starting from it distributed as coalescing Brownian motion a.s.,. It is enough to verify condition (2) of Theorem 17. Let \( X' \) be the closure of the paths of \( X \) starting from \( D \). So, by construction, \( X' \) has the distribution of the Brownian web, \( \mu \).

Consider \( p + 1 \) equally spaced out points \( z_j = (a + \frac{j(b-a)}{p}, t_0) \) in \([a,b]\) with \( j = 0, 1, ..., p \). Let \( \eta_p = |\{ x \in \mathbb{R} \mid \text{there exists a path in } X \text{ touching both a point in } \{z_0, ..., z_p\} \text{ and } (x, t + t_0)\}| \)

and \( \eta'_p \) defined similarly. Evidently \( \eta_X \geq \eta_p \) and \( \eta_{X'} \geq \eta'_p \).

From \( (B'_1) \eta_p = \eta'_p \) a.s., recalling that \( X \) has a unique path starting at every point and so has \( X' \).

Let \( \epsilon = \frac{b-a}{p} \), \( 1 \leq j \leq p \) and let \( a'_j = \frac{z_{j-1} + z_j}{2} \). Put \( A_j = \{ N_{t_0,t}(\{a'_j - \epsilon, a'_j + \epsilon\}) \neq (N_{t_0,t}^+(\{a'_j - \epsilon, a'_j + \epsilon\}) \cup N_{t_0,t}^-(\{a'_j - \epsilon, a'_j + \epsilon\})) \}) \). We use \( (B'_2) \), which implies \( \mathbb{P}(X^{-1}(A_j)) = \mu(A_j) \to 0 \) as \( \epsilon \to 0^+ \). Therefore, as \( p \to \infty \), noting \( \{ \eta_X > \eta_p \} = \bigcup_{j=1}^p \{ X^{-1}(A_j') \} \), \( \mathbb{P}(X^{-1}(A_j)) = \mathbb{P}(\{ \eta_X > \eta_p \}) \to 0 \) as \( p \to \infty \). Thus \( \mathbb{P}(\eta_X > \eta_p) = 0 \) because \( \{ \eta_X > \eta_p \} \subseteq \{ \eta_X > \eta'_p \} \) for all \( p \). Therefore \( \mathbb{P}(\eta_X \geq k) \leq \mathbb{P}(\eta_{X'} \geq k) \). Hence condition (2) of theorem 17 holds, so \( \mu \) is the distribution of the Brownian web. \( \square \)

The remainder of the proof of Theorem 23 consists in showing that \( \mu \) satisfies condition (1) of Theorem 17 and \( (B'_1), (B'_2) \). It is clear that \( (B'_1) \) together with (I) imply condition (1) of Theorem 17, that is in \( X \) there is, from any point in \( D \) a coalescing Brownian motion starting. The verification of \( (B'_1) \) and \( (B'_2) \) from \( (B_1) \) and \( (B_2) \) is technical, one uses the remaining features of the estimates \( (B_1) \) and \( (B_2) \) and set properties of weak covengence. \( (B'_2) \) is established via an intermediate step of estimating how close various paths come to a point \( a \) that is approximated. We could do no more than repeat the details, so we skip this part of the proof, but the reader is strongly invited to look at [12, p. 34-37].

### C.2 Verification of (I) in the proof of Theorem 28

We maintain the notation of Section 3.4 and wish to prove Lemma 31, which needs some more steps. Let \( x^1, x^2 \in \mathbb{R}^2 \) and consider the random walks \( \xi^{x^1}(t) \) and \( \xi^{x^2}(t) \).
Recall that by the work of Ferrari, Landim and Thorisson [9] we know that a.s. at some time $t_0 \geq x_1^2 \lor x_2^2$ these will meet and coalesce. As a corollary to that fact we have

**Lemma 44 (Lemma 2.4. of [8])** For any $\lambda > 0, r > 0$,

$$\mathbb{P}_x(\sup_{t \geq x_2} |\xi^r_1(t) - \xi^r_1(t)| \geq \sigma) \to 0$$

as $\sigma \to \infty$ for any $x, y \in \mathbb{R}^2$ with $x_2 = y_2$.

Now for $y^1, \ldots, y^m \in D$, consider $(\tilde{\xi}^y_\delta, \ldots, \tilde{\xi}^y_m)$ as a random element in $\Pi^m$. Ferrari et al. prove Lemma 31 by defining a ‘$\delta$-coalescence function’ $f_\delta$ on $\Pi^m$ (we follow the informal treatment in [8, 854-55] for the painstaking details see [7, 9-12]), to get around the obstacle that paths are not completely independent up to coalescence in the Poisson tree.

Let $\gamma_{\delta, 0} = \text{min}(y^1_2, \ldots, y^m_2)$, assume $\delta > 0$ is so close to 0 that $-\infty < \gamma_{\delta, 0} < \gamma_{\delta, 1} < \ldots < \gamma_{\delta, m-1} < \infty$ where, for $1 \leq k \leq m - 1$, $\gamma_{\delta, k}$ is the time when the $k$th $\delta$-coalescence occurs. Here we mean by a ‘$\delta$-coalescence’ at time $t_0$ that at that time two particles get within distance smaller than $2\sqrt{3}\delta = 2r\delta$. Then we renew the system by coalescing the two particles into the left one, specifying a continuous path for both particles. Denote the set of paths resulting from the $m - 1$ coalescences by $f_\delta(\tilde{\xi}^y_\delta, \ldots, \tilde{\xi}^y_m)$.

As in Section 1.3, define from $m$ independent Brownian motions $B^{y^1}, \ldots, B^{y^m}$, starting at $y^1, \ldots, y^m$, a set of $m$ coalescing Brownian motions and denote it by $f(B^{y^1}, \ldots, B^{y^m})$.

Finally, suppose $\hat{\xi}^y_\delta$ has the same distribution as $\tilde{\xi}^y_\delta$, and as a random element in $\Pi^m$, $(\hat{\xi}^y_\delta, \ldots, \hat{\xi}^y_m)$ has independent components. So we have, in particular, not assumed coalescence of the $(\tilde{\xi}^y_\delta, \ldots, \tilde{\xi}^y_m)$. It remains to prove one more lemma to verify (I).

**Lemma 45 (Lemma 2.5. of [8])** Let $(\hat{\xi}^y_\delta, \ldots, \hat{\xi}^y_m)$ be $m$ continuous random walks as defined in Section 3.4 and let $(\tilde{\xi}^y_\delta, \ldots, \tilde{\xi}^y_m)$ have independent components with each $\hat{\xi}^y_\delta$ having the same distribution as $\tilde{\xi}^y_\delta$ for all $1 \leq i \leq m$. Then

(a) $f_\delta(\hat{\xi}^y_\delta, \ldots, \hat{\xi}^y_m)$ has the same distribution as $f_\delta(\tilde{\xi}^y_\delta, \ldots, \tilde{\xi}^y_m)$,

(b) $f_\delta(\tilde{\xi}^y_\delta, \ldots, \tilde{\xi}^y_m)$ converges in distribution as $\delta \to 0$ to $f(B^{y^1}_1, \ldots, B^{y^m}_1)$,

(c) for any $\epsilon > 0$, $\mathbb{P}(d^m(f_\delta(\tilde{\xi}^y_\delta, \ldots, \tilde{\xi}^y_m), (\hat{\xi}^y_\delta, \ldots, \hat{\xi}^y_m)) \geq \epsilon) \to 0$, as $\delta \to 0$.

Remark: Note that Lemma 45 immediately implies, by the fact that convergence in probability implies convergence in distribution, Lemma 31, and so one has verified (I).

Idea of Proof:

(a) is immediate from the definition of $f_\delta$.

For (b) we know that by lemma 29 and independence $(\hat{\xi}^y_\delta, \ldots, \hat{\xi}^y_m)$ converges in distribution to $(B^{y^1}_1, \ldots, B^{y^m}_1)$ as $\delta \to 0$.
C.3 Verification of \((B'_2)\) in the proof of Theorem 28

One can use a particular extended continuous mapping theorem for weak convergence that holds by properties of \(f, f_\delta\) and \(d^m\) to prove (b); see [7, p. 13] and the references therein for details.

(c) is a complicated estimate; a key ingredient is to note that, although it may be that \(f_\delta(\xi_\delta^1, \ldots, \xi_\delta^j)\) and \((\xi_\delta^1, \ldots, \xi_\delta^j)\) do not agree for some time, eventually they will follow the same paths in any random realisation (so applying Lemma 44), and to exploit that \(\tanh\) is Lipschitz (again see [7, p. 13] for details).

C.3 Verification of \((B'_2)\) in the proof of Theorem 28

We maintain the notation of Section 3.4. To calculate the desired bound on \(\limsup_{n \to \infty} \mathbb{P}(\eta_{X_{n_t}}(0, t, 0, \epsilon) \geq 3)\), it is, by the rescaling, sufficient to verify that

\[
\frac{1}{\epsilon} \limsup_{N \to \infty} \mathbb{P}(\eta_{X_t}(0, tN, 0, \epsilon \sqrt{N}) \geq 3) \to 0 \text{ as } \epsilon \to 0^+ ,
\]  

(C.1)

where \(X_t\) is the unrescaled Poisson web, as in Defintion 27. To this end fix \(t > 0\) and we will condition the probability in (C.1) on the set of points of intersection of \(\{\xi^s, s \in S\}\) with \([0, \epsilon \sqrt{N}]\), in increasing order. Denote those by \(\{K_1, \ldots, K_L\}\) where \(L, K_1, \ldots, K_L\) are random variables, \(L\) finite and note that by definition of the Poisson tree no two distinct \(K_i\) can have distance smaller than \(r\).

Choose points \(\{x_1, \ldots, x_n\} \subset [0, \epsilon \sqrt{N}\)\), let \(\xi_j := \xi(x_j, 0), 1 \leq j \leq n\) and \(\eta' = \eta'(x_1, \ldots, x_n) = |\{\xi_j(tN) : 1 \leq j \leq n\}|\). It is then evident by the Markov property of the random walks that if \(L = n\) and \(K_1 = x_1, \ldots, K_n = x_n\), then the probability in (C.1) equals \(\mathbb{P}(\eta' \geq 3)\).

We obtain an upper bound for that probability by enlarging the set \(\{x_1, \ldots, x_n\}\), if necessary by the endpoints 0 and \(\epsilon \sqrt{N}\) and intermediate points to ensure that for all \(j, r \leq x_j - x_{j-1} \leq 2r\). This also ensures that \(n \leq \epsilon \sqrt{N}/r + 1\) and the enlargement only increases the probability we want to bound. The crux of the argument is the simple observation that if \(\eta' \geq 3\), then there should be some \(2 \leq j \leq n - 1\) such that \(\xi_{j-1}(tN) < \xi_j(tN) < \xi_n(tN)\). Therefore

\[
\mathbb{P}(\eta' \geq 3) \leq \sum_{j=2}^{n-1} \mathbb{P}(\xi_{j-1}(tN) < \xi_j(tN) < \xi_n(tN))
\]

\[
= \sum_{j=2}^{n-1} \int_{\Pi_j} \mathbb{P}(\xi_{j-1}(tN) < \xi_j(tN) < \xi_n(tN) | \xi_j = \pi) d\mu_{\xi_j}
\]

\[
= \sum_{j=2}^{n-1} \int_{\Pi_j} \mathbb{P}(\xi_{j-1}(tN) < \xi_j(tN) | \xi_j = \pi) \mathbb{P}(\xi_j(tN) < \xi_n(tN) | \xi_j = \pi) d\mu_{\xi_j} , \quad (C.2)
\]

where \(\Pi_j \subseteq \Pi\) is the state space of \(\xi_j\) and \(\mu_{\xi_j}\) its distribution. The second equality follows by independence of the path to the left and that to the right of \(j\) conditioned on \(\xi_{\xi_j} = \pi\).
C.3 Verification of \((B''_2)\) in the proof of Theorem 28

We now use Lemma 33, the FKG-property of \(\mu_j\), in the version of an increasing and a decreasing function (see [16, p. 27]), and claim that \(\mathbb{P}(\xi_{j-1}(tN) < \xi_j(tN) \mid \xi_j = \pi)\) decreases and \(\mathbb{P}(\xi_j(tN) < \xi_n(tN) \mid \xi_j = \pi)\) increases in \(\pi\), see the discussion in [7, p.19]. So (C.2) is bounded above by

\[
\sum_{j=2}^{n-1} \int_{\Pi_j} \mathbb{P}(\xi_{j-1}(tN) < \xi_j(tN)|\xi_j = \pi) d\mu_{\xi_j} \cdot \int_{\Pi_j} \mathbb{P}(\xi_j(tN) < \xi_n(tN)|\xi_j = \pi) d\mu_{\xi_j}
\]

\[
= \sum_{j=2}^{n-1} \mathbb{P}(\xi_{j-1}(tN) < \xi_j(tN)) \mathbb{P}(\xi_j(tN) < \xi_n(tN))
\]

\[
\leq \mathbb{P}(\xi_0(tN) < \xi_n(tN)) \sum_{j=2}^{n-1} \mathbb{P}(\xi_{j-1}(tN) < \xi_j(tN))
\]

where the last inequality follows since evidently \(\mathbb{P}(\xi_j(tN) < \xi_n(tN))\) is non-increasing in \(j\). Now the probabilities inside the sum are bounded above by \(\mathbb{P}(\xi^{(0,0)}(tN) < \xi^{(2r,0)}(tN))\), and so we get

\[
\mathbb{P}(\eta' \geq 3) \leq (n - 1) \mathbb{P}(\xi^{(0,0)}(tN) < \xi^{(2r,0)}(tN)) \mathbb{P}(\xi^{(0,0)}(tN) < \xi^{(r\sqrt{N},0)}(tN))
\]

\[
\leq \frac{\epsilon\sqrt{N}}{r} \mathbb{P}(T > tN) \mathbb{P}(T_{e,N} > tN).
\]

The last inequality follows from the bound on \(n\) obtained earlier. \(T\) is, as introduced in Lemma 34, the time when \(\xi^{(0,0)}\) and \(\xi^{(2r,0)}\) coalesce, and \(T_{e,N}\) is the analogous time for \(\xi^{(0,0)}\) and \(\xi^{(r\sqrt{N},0)}\).

Recall that we are trying to estimate the \(\limsup_{N \to \infty}\) of the right-hand side of the preceding formula. Because in the limit, the distributions of the random walks become those of Brownian motions, we have that

\[
\limsup_{N \to \infty} \mathbb{P}(T_{e,N} > tN) = \mathbb{P}(T_{e,B} > t),
\]

where \(T_{e,B}\) denotes the time at which two i.i.d. Brownian motions starting at the same time distance \(\epsilon\) apart meet and coalesce. By the calculation in the proof of Theorem 24 we know that for \(t\) fixed this is of order \(O(\epsilon)\). By the previous lemma, we have \(\mathbb{P}(T > tN) \leq \frac{\epsilon}{\sqrt{N}}\). Together these estimates imply that

\[
\limsup_{N \to \infty} \mathbb{P}(\eta_{X_1}(0, tN, 0, \epsilon\sqrt{N}) \geq 3) \leq \limsup_{N \to \infty} \frac{\epsilon\sqrt{N}}{r} \mathbb{P}(T_{e,N} > tN) \mathbb{P}(T > tN) = C\epsilon^2
\]

for some positive constant \(C\) not depending on \(N\), so we have proved that \(\limsup_{N \to \infty} \mathbb{P}(\eta_{X_1}(0, tN, 0, \epsilon\sqrt{N}) \geq 3)\) is of order \(o(\epsilon)\) as \(\epsilon \to 0\) and so have verified \((B''_2)\). \(\Box\)

So, whereas in Section 3.2, the verification of \((B''_2)\) followed by a stronger FKG-property that allowed it to be verified with \((B''_1)\) at once, for the Poisson tree verifying \((B''_2)\) turns out to be much more subtle.

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Appendix D

Latest developments: coalescence with branching

The purpose of this last appendix is to give the reader some idea of the latest developments concerning the Browian web, as many more publications on the topic have appeared in the last few years (and particularly in 2007) than we have been able to discuss in detail. Consequently we shall be eclectic rather than rigorous in this chapter and only sketch ideas instead of giving details.

Some work has been done relating the Brownian web to the theory of stochastic flows: Notably Tsirelson discussed the Brownian web in his St. Flour lecture notes [34] and in [33] in this context. Recently Howitt and Warren [18] published on perturbing the system of simple coalescing random walks by making the increment dependent on a second (continuous) time parameter $s$ and they consider the resulting ‘dynamics’ for the Brownian web.

Fontes, Newman, Ravishankar and Schertzer have also recently written on this configuration (see [15]) which they call dynamical discrete web and consider exceptional random values of $s$ where the paths do not behave like ordinary random walk paths. A paper of Newman, Ravishankar and Schertzer (“Marking (1,2) points of the Brownian web and applications”) containing more about the scaling limit is in preparation.

Another paper of Fontes, Isopi et al. [10] goes back to the relationship between the voter model and the Brownian web and considers how small noise in the voter model changes the continuum limit. Yet a different connection with the Brownian web is found in the work of Baccelli and Bordenave [3] who consider further random graphs based on a homogeneous Poisson process and generate open problems of whether those graphs converge to the Brownian web as well (see [3, p. 25 and 48]).

We now wish to focus on a particular extension of the results we presented earlier in some more detail: The problem of what can be said about the scaling limit of coalescing random walks when a small probability of branching is added. Sun and Swart, in the very substantial paper [31] (and in “special points of the Brownian net” (in preparation)) study the scaling limit of the following collection of paths.

Consider again all ‘even’ points of discrete space-time and fix a branching probability $\beta$. For each $(m, n)$ with $m + n$ even let $Y_{m,n}$ be the ‘increment’ that there is a path
from \((m, n)\) to \((m - 1, n + 1)\) with probability \(\frac{1 - \beta}{2}\), from \((m, n)\) to \((m + 1, n + 1)\) with probability \(\frac{1 - \beta}{2}\), and with probability \(\beta\) a path to both \((m - 1, n + 1)\) and \((m + 1, n + 1)\).

Figure 6: Coalescing-branching random walks in discrete time. Source: [31, p. 3], slightly modified.

Denote the resulting system of coalescing-branching random paths by \(U_\beta\). Rescale
space-time by a factor \(\epsilon\) in space and \(\epsilon^2\) as usual. Of course the case \(\beta = 0\) gives as a
scaling limit the Brownian web. Sun and Swart examine what can be said about the
scaling limit if \(\epsilon\) and \(\beta\) tend to 0 simultaneously. They prove that if \(\beta \epsilon \to b\) for \(b \geq 0\),
then in (almost) the same topology introduced in this dissertation \(U_\beta\) converges in
distribution to a random object \(N_b\), called Brownian net (with branching parameter
\(b\), where \(b = 0\) also gives the Brownian web). More precisely:

**Theorem 46 (Theorem 1.1. of [31])** Denote by \(S_\epsilon(U_\beta)\) the rescaled collection of
coalescing-branching random walks. Then there exist \((H, \mathcal{F}_H)\)-valued random variables
\(N_b\) such that if \(\epsilon_n, \beta_n \to 0\) and \(\frac{\beta_n}{\epsilon_n} \to b \geq 0\), then \(S_{\epsilon_n}(U_{\beta_n}) \to N_b\) in distribution.
Moreover if \(\mathcal{L}(\cdot)\) denotes distribution, then for \(\epsilon, b > 0\), \(\mathcal{L}(S_\epsilon(N_b)) = \mathcal{L}(N_b)\) and
\(N_b \neq N_0 = W\), where \(W\) is the Brownian web.

Sun and Swart discuss in [31] a variety of methods of characterising this new object,
convergence to it and its relationship with the Brownian web.

We can do no more then say a few words about one way of characterizing the
object: What makes the Brownian net more complicated than the Brownian web is
that for any space-time point \(x = (x_1, x_2) \in \mathbb{R}^2\) there are many paths, call their set
\(N(x)\), starting from it. The key to the characterisation of the Brownian net is that
one can however still identify a.s. a well-defined left-most path and a right-most path
\(l_x, r_x \in N(x)\) in the sense that \(l_x(s) \leq \pi(s) \leq r_x(s)\) for any \(s \geq x_2\), \(\pi \in N(x)\) (see
[31, p. 7]).

Two ingredients are needed for a characterisation theorem: first, the distribution
of a finite number of left-most and right-most paths
\((l_z, \ldots, l_{z^k}, r_{z^l}, \ldots, r_{z^l})\) for deterministic \(z_1, \ldots, z_k, z'_1, \ldots, z'_k \in \mathbb{R}^2\) needs to be characterised.
It turns out that the \(k\) left-most and \(k'\) right-most paths are distributed as
coalescing Brownian motions with drift 1 to the left, and to the right respectively, and
paths evolve independently as long as they do not coincide.
In fact (see [31, p. 7]), it suffices to characterise the joint evolution between one left-most and one right-most path \( l_z = l(x,s) \) and \( r_z' = r(x',s') \). Sun and Swart provide for this the two-dimensional *left-right SDE*

\[
\begin{align*}
\text{d}L_t &= \mathbf{1}_{\{L_t \neq R_t\}} \text{d}B_t^L + \mathbf{1}_{\{L_t = R_t\}} \text{d}B_t^R - \text{d}t \\
\text{d}R_t &= \mathbf{1}_{\{L_t \neq R_t\}} \text{d}B_t^R + \mathbf{1}_{\{L_t = R_t\}} \text{d}B_t^L + \text{d}t
\end{align*}
\]

of which a unique weak solution characterises the joint evolution after time \( s \lor s' \). Here \( B_t^L, B_t^R, B_t^S \) are independent Brownian motions and \( L_t, R_t \) subject to the constraint that \( L_t \leq R_t \) if \( t \geq \inf \{ u \geq s \lor s' : L_u \leq R_u \} \). Call a system \((l_{z_1},...,l_{z_k},r_{z_1}',...,r_{z_k}')\) thus distributed ‘left-right coalescing Brownian motions’.

Second, one also needs to characterise all paths, not just left- and right-most paths, and Sun and Swart do this by looking at concatenating pieces of paths at times when they coincide and call this ‘hopping’. Define \( t \) to be an intersection time of two paths \( \pi_1, \pi_2 \in \Pi \) with starting times \( \sigma_1, \sigma_2 \) if \( \infty > t > \sigma_1 \lor \sigma_2 \) and \( \pi_1(t) = \pi_2(t) \); and \( t \) is a crossing time if paths cross and do not just touch at that time.

For \( A \subset \Pi \) let \( \text{Hop}_{\text{cross}}(A) \) be the smallest set of paths closed under hopping at crossing times (for the formal definition see [31, p. 7]). Then we can characterise the Brownian net as follows:

**Theorem 47 (Theorem 1.3. of [31])** There exists a \((H, \mathcal{F}_H)\)-valued random variable \( N \), the Brownian net, whose distribution is uniquely determined by the following conditions.

1. For each deterministic \( x \in \mathbb{R}^2 \) a.s. \( N(x) \) contains a unique left-most and right-most path, \( l_x \) and \( r_x \).

2. For any deterministic set of points \( z_1, ..., z_k, z_1', ..., z_k' \in \mathbb{R}^2 \), the collection of paths \((l_{z_1}, ..., l_{z_k}, r_{z_1}', ..., r_{z_k}')\) is distributed as a collection of left-right coalescing Brownian motions, as defined above.

3. For any deterministic countable dense sets \( D^L, D^R \subset \mathbb{R}^2 \)

\[
N = \overline{\text{Hop}_{\text{cross}}(\{l_z : z \in D^L\} \cup \{r_z : z \in D^R\})}
\]

Unfortunately we cannot pursue this fascinating extension to the theory of the Brownian web here, but refer the reader in particular to the stimulating discussion concerning its relevance in [31, p. 14-16].

Instead we mention in passing another paper of Fontes and Newman [14] who also extend the Brownian web by adding the possibility of branching. Certain one-dimensional stochastic flows can be shown to converge to coalescing Brownian motions, but it turns out that their scaling limit will need to contain bifurcation as well as coalescence. Fontes and Newman call it the ‘full Brownian web’. It is based on the double Brownian web and characterised as a collection of paths so that starting from every space-time point there is both a forward and a backward path starting. The paths are non-crossing, but may touch, coalesce and bifurcate, and are such that the
Finally, a survey of the latest developments on the Brownian web would not be complete without mentioning recent work of Turner [35] and Norris and Turner [23], who look at the convergence of a class of stochastic flows on the circle to the Brownian web. Contrarily to most writers on the subject who seemed to have accepted the viewpoint of Fontes et al. to treat the Brownian web as a random variable in $H$, – but in line with Tsirelson in [34]) – Turner and Norris take the Brownian web as an element of a space of stochastic flows on the circle. Instead of using a compactification of $\mathbb{R}^2$ as in our Section 2.1, their Brownian web ‘lives’ on the circle, and on a finite time interval $[-T,T]$. Moreover, Turner’s ([35, p. 79]) and Norris and Turner’s ([23, p. 10]) characterisation theorem for the Brownian web does not need an additional minimality property in the sense of condition (3) in Theorem 9. Norris and Turner are able to show that for their choice of space for the Brownian web there is a unique measure for which the finite-dimensional distributions of the stochastic flows are those of coalescing Brownian motions (which in particular simplifies convergence issues) and provide an isomorphism onto a space related to that of the Brownian web and full Brownian web, to show that their construction of the Brownian web is essentially equivalent to that of Fontes et al.. In particular they believe that “working in a space whose structure inherently contains the restrictions imposed by the Brownian web is more natural” and they “find that this simplifies characterisation and convergence results” ([23, p. 2]) as well as saves to deal with issues of tightness.

We do not feel to be in the position to comment on whether the approach of Fontes, Newman et al. or that of Norris and Turner to the Brownian web is the more suitable and useful, as a thorough comparison would require another dissertation. We merely wish to note that there is hence ongoing debate on how to characterise the Brownian web in a mathematically most fruitful way.

Even if the preceding exposition on extensions of the Brownian web is too sketchy in parts, it is hoped that we achieved to highlight that the Brownian web is really at the frontline of research in probability theory. It should also have become clear why we chose to base our discussion of the Brownian web on the publications of Fontes et al. [12] and Ferrari et al. [8].
Bibliography


